The Ergodic Hierarchy of Mixing, van der Corput's difference theorem, and the ergodic theory of noncommuting operators

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Overview

1. Review
   - Van der Corput's difference theorem and some applications
   - Mixing properties
2. Mixing van der Corput difference theorems
3. Ergodic theorems for noncommuting operators
   - Background
   - New results from mixing vdCs
4. Examples of systems with singular spectrum

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The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is **uniformly distributed** if for any open interval $(a,b) \subseteq [0,1]$ we have

$$\lim_{N\to\infty} \frac{1}{N} |\{1 \le n \le N \mid x_n \in (a,b)\}| = b - a. \tag{1}$$

Theorem (van der Corput, [27])

If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha)_{n=1}^{\infty}$ is uniformly distributed.

Hilbertian van der Corput Difference Theorems 1/3

Theorem (HvdCDT1, [3, Theorem 1.4])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0, \tag{2}$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|\right| = 0. \tag{3}$$

Hilbertian van der Corput Difference Theorems 2/3

Theorem (HvdCDT2, [3, Page 3])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \tag{4}$$

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\| = 0. \tag{5}$$

Theorem (HvdCDT3, [3, Theorem 1.5], or [18, Lemmas 4.9, 7.5])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \tag{6}$$

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|\right| = 0. \tag{7}$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \quad (8)$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy)

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \quad (9)$$

Furstenberg's proof in [17, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of $(n^2\alpha)_{n=1}^{\infty}$. See also [4, Theorem 2.1] for a proof using HvdCDT1 directly.

Applications of HvdCDTs 2/2

Theorem (Furstenberg, [17])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$*, any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$*, and any* $\ell \in \mathbb{N}$*, there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A\right) > 0. \tag{10}$$

The proof presented in [9] uses HvdCT3 as Theorem 7.11, and the proof in [18] uses a variation.

Theorem (Bergelson and Leibman, [6])

For any m.p.s. $(X, \mathscr{B}, \mu, \{T_i\}_{i=1}^{\ell})$ *with the* T_i*s commuting, any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$*, and any* $\{p_i(x)\}_{i=1}^{\ell} \subseteq x\mathbb{N}[x]$*, there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \cdots \cap T_\ell^{-p_\ell(n)}A\right) > 0. \tag{11}$$

Uses an equivalent form of HvdCT3 as Lemma 2.4.

Some of the Ergodic Hierarchy of Mixing

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If for every $f, g \in L^2_0(X, \mu)$

1. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle U_T^n f, g\rangle = 0$, then $\mathcal{X}$ is ergodic.
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U_T^n f, g\rangle| = 0$, then $\mathcal{X}$ is weakly mixing,
3. $\lim_{n\to\infty} \langle U_T^n f, g\rangle = 0$, then $\mathcal{X}$ is strongly mixing,
4. and if $L^2_0(X, \mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n,m\in\mathbb{Z}}$, then $\mathcal{X}$ has Lebesgue spectrum.
5. which is the same as $(\langle U_T^n f, g\rangle)_{n=1}^{\infty}$ being Fourier coefficients of some $h \in L^1([0,1], \mathcal{L})$, where $\mathcal{L}$ is the Lebesgue measure.

These definitions also apply to individual elements $f \in L^2_0(X, \mu)$.

The Symmetric Ergodic Hierarchy of Mixing

Theorem

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ *be a m.p.s. If for every* $f \in L^2_0(X, \mu)$

1. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle U_T^n f, f\rangle = 0$, *then* $\mathcal{X}$ *is ergodic,*
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U_T^n f, f\rangle| = 0$, *then* $\mathcal{X}$ *is weakly mixing,*
3. $\lim_{n\to\infty} \langle U_T^n f, f\rangle = 0$, *then* $\mathcal{X}$ *is strongly mixing,*
4. $\mathcal{X}$ *has Lebesgue spectrum if* $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ *are the Fourier coefficients of some* $h \in L^1([0,1], \mathcal{L})$ *taking nonnegative real values.*

This theorem also applies to individual elements $f \in L^2_0(X, \mu)$.

Dual notions to various levels of mixing

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If $f \in L^2(X, \mu)$ satisfies

1. $U_T f = f$, then f is invariant.
2. $f \in L^2(X, K, \mu)$ where (X, K, μ, T) is the Kronecker factor of $(X, \mathscr{B}, \mu, T)$, then f is compact.
3. $f \in L^2(X, \mathscr{B}_P, \mu)$, where $\mathscr{B}_p$ is the Parreau factor from [22, Theorem 11], then f is 'anti-mixing' (provisional term).
4. If $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ are the Fourier coefficients of a measure $\mu_{f,T}$ that is mutually singular with the Lebesgue measure then f has singular spectrum.
5. T has singular spectrum if all $f \in L^2(X, \mu)$ have singular spectrum, i.e., the maximal spectral type of T is singular.

Disjointness and orthogonality

Theorem

For $f, g \in L^2_0(X, \mu)$, we have $\langle f, g \rangle = 0$ if

1. *f is invariant and g is ergodic.*
2. *f is compact and g is weakly mixing.*
3. *f is 'anti-mixing' and g is strongly mixing.*

\# *f has singular spectrum and g has Lebesgue spectrum.*

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A weak mixing van der Corput difference theorem

Theorem (MvdCT3)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{12}$$

then $(x_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence. This means that for any other bounded sequence $(y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ we morally (but not literally) have that

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, y_n \rangle \right| = 0. \tag{13}$$

Loosely speaking, this can be interpretted as a weak mixing in any ultrapower $\mathscr{H}$ of $\mathcal{H}$ with respect to a unitary operator induced by the left shift. Note that elements of $\mathscr{H}$ are sequences in $\mathcal{H}$.

A strong mixing van der Corput difference theorem

Theorem (MvdCT2)

If $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{14}$$

then $(x_n)_{n=1}^\infty$ is a nearly strongly mixing sequence. This means that for any other bounded sequence $(y_n)_{n=1}^\infty \subseteq \mathcal{H}$ we morally (but not literally) have that

$$\lim_{h\to\infty} \left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, y_n \rangle \right| = 0. \tag{15}$$

Loosely speaking, this can be interpretted as a strong mixing in any ultrapower $\mathscr{H}$ of $\mathcal{H}$ with respect to a unitary operator induced by the left shift. Note that elements of $\mathscr{H}$ are sequences in $\mathcal{H}$.

A Lebesgue spectrum vdCdt

Theorem (MvdCT1)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying for all $h \in \mathbb{N}$

$$\sum_{h=1}^{\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \tag{16}$$

then $(x_n)_{n=1}^{\infty}$ is a spectrally Lebesgue sequence. In particular, if $\mathcal{H} = L^2(X,\mu)$ and $(y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is spectrally singular, then we have

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n y_n \right\| = 0. \tag{\#}$$

Upgrading the weak convergence from # to the strong convergence in # necessitates a new proof of the classical vdCDT. See [10, Chapter 2] for variations of MvdCT related to other levels of mixing, as well as uniform distribution. See also [26].

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Noncommutative ergodic theorems 1/2

Theorem ([12, Corollary 1.7])

Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{t\to\infty} \frac{a(t)}{t^{d+\epsilon}} = \lim_{t\to\infty} \frac{t^{d+1}}{a(t)} = \infty. \quad \left(e.g.\ a(t) = t^{1.5}\right) \tag{17}$$

Furthermore, let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving transformations. Suppose that the system $(X, \mathscr{B}, \mu, T)$ has zero entropy. Then

(i) *For every $f, g \in L^\infty(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f|\mathcal{I}_T] \cdot \mathbb{E}[g|\mathcal{I}_S], \tag{18}$$

where the limit is taken in $L^2(X, \mu)$.

Noncommutative ergodic theorems 2/2

Theorem (Continued)

(ii) *For every $A \in \mathscr{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap T^{-n}A \cap S^{-\lfloor a(n) \rfloor}A\right) \geq \mu(A)^3. \qquad (19)$$

In [13] a similar theorem is proven for $a(n) = p(n)$ with $p(x) \in \mathbb{Z}[x]$ of degree at least 2. The zero entropy assumption on T cannot be removed as seen by [18, Page 40] or [2, Example 7.1]. In [24] it was shown that every T with singular spectrum must also have zero entropy. Note that the Horocycle flow has zero entropy [19] and Lebesgue spectrum [23]

There is no Roth Theorem for solvable groups

Theorem ([7, Theorem 1.2])

Let G be a finitely generated solvable group of exponential growth. For any partition $R \bigcup P = \mathbb{Z} \setminus \{0\}$, there exist an action $\{T_g\}_{g \in G}$ of G on a probability space $(X, \mathscr{B}, \mu)$, $g_1, g_2 \in G$, and set $A \in \mathscr{B}$ with $\mu(A) > 0$ such that

$$\mu\left(T_{g_1^n}A \cap T_{g_2^n}A\right) = 0 \quad \textit{if } n \in R \textit{ and}$$
$$\mu\left(T_{g_1^n}A \cap T_{g_2^n}A\right) \geq \frac{1}{6} \quad \textit{if } n \in P.$$

Note that the group used in [2, Example 7.1] is non-solvable.

Another Example

Theorem ([15, Lemma 4.1])

Let $a, b : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ *be injective sequences and* F *be any subset of* $\mathbb{N}$*. Then there exist a probability space* $(X, \mathscr{B}, \mu)$*, measure preserving automorphisms* $T, S : X \to X$*, both of them Bernoulli, and* $A \in \mathscr{B}$*, such that*

$$\mu\left(T^{-a(n)}A \cap S^{-b(n)}A\right) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \tag{20}$$

Application 1/4

Theorem (F., 2022)

Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving automorphisms for which T has singular spectrum. Let $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed in the orbit closure of α for all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$.

1. *For any $f, g \in L^\infty(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{k_n} g = \mathbb{E}[f|\mathcal{I}_T]\,\mathbb{E}[g|\mathcal{I}_S], \tag{21}$$

with convergence taking place in $L^2(X, \mu)$.

Application 1/4 continued

Theorem (Continued)

(ii) *If $A \in \mathscr{B}$ then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap T^{-n}A \cap S^{-k_n}A\right) \geq \mu(A)^3. \qquad (22)$$

(iii) *If we only assume that $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$, then (i) and (ii) hold when S is totally ergodic.*

Examples include $k_n = \lfloor a(n) \rfloor$ with $a(n)$ being as in frame 19, $k_n = \lfloor n^2 \log^2(n) \rfloor$, and for part (iii) we may take $k_n = p(n)$ for $p(x) \in x\mathbb{Z}[x]$ with degree at least 2.

Sets of K but not $K+1$ recurrence?

Theorem ([14, Theorem 1.4 and Corollary 4.4])

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$.

(i) *If $(X, \mathscr{B}, \mu)$ is a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which*

$$\mu\left(A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{23}$$

(ii) *There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A\right) = 0. \tag{24}$$

Application 2/4

Theorem (F., 2022)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{ n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right] \right\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ commuting measure preserving automorphisms. Let $T : X \to X$ be an measure preserving automorphism with singular spectrum, and for which $\{T, S_1, S_2, \cdots, S_{k-1}\}$ generate a nilpotent group. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu\left(A \cap T^{-n}A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{25}$$

Since the system $(\mathbb{T}^2, \mathscr{B}^2, \mathcal{L}^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ can be used in item (ii) of the last slide when $k = 2$, the current theorem does not hold for a general T with 0 entropy. Also note that the maximal spectral type of T is $\mathcal{L} + \delta_\alpha$.

Application 3/4 (A special case)

Theorem (F., 2022)

Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving automorphisms. Suppose that T has singular spectrum and S is totally ergodic. Let $p_1, \cdots, p_K \in \mathbb{Q}[x]$ be integer polynomials for which $deg(p_1) \geq 2$ and $deg(p_i) \geq 2 + deg(p_{i-1})$. For any $f, g_1, \cdots, g_K \in L^\infty(X, \mu)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \prod_{i=1}^{K} S^{p_i(n)} g_i = \mathbb{E}[f|\mathcal{I}_T] \prod_{i=1}^{K} \int_X g_i d\mu, \tag{26}$$

with convergence taking place in $L^2(X, \mu)$.

Application 4/4 (A special case)

Theorem (F., 2022)

Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, R, S : X \to X$ be measure preserving automorphisms. Suppose that T has singular spectrum, R and S commute, and S is weakly mixing. Let $\ell \in \mathbb{N}$ and let $p_1, \cdots, p_\ell \in \mathbb{Q}[x]$ be pairwise essentially distinct integer polynomials, each having degree at least 2. For any $f, h, g_1, \cdots, g_\ell \in L^\infty(X, \mu)$ satisfying $\int_X g_j d\mu = 0$ for some $1 \le j \le \ell$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot R^n h \cdot \prod_{j=1}^{\ell} S^{p_j(n)} g_j = 0, \tag{27}$$

with convergence taking place in $L^2(X, \mu)$.

An example to justify our assumptions

Consider the m.p.s. $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T, S)$ with $S(x,y) = (x+2\alpha, y+x)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $T(x,y) = (x, y+x)$. We see that $([0,1]^2, \mathscr{B}, \mathcal{L}^2, S)$ and $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T)$ are both zero entropy systems that are not weakly mixing, and the former is totally ergodic. Furthermore, T and S generate a 2-step nilpotent group. For $f_0(x,y) = e^{2\pi i(x-y)}$, $f_1(x,y) = e^{2\pi iy}$, and $f_2(x,y) = e^{-2\pi ix}$, we see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^n f_0(x,y) S^n f_1(x,y) S^{\frac{1}{2}(n^2-n)} f_2(x,y)$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\left((1-n)x-y+y+nx+(n^2-n)\alpha-x-(n^2-n)\alpha\right)} = 1 \neq 0.$$

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Examples of systems with singular spectrum

In [5, Proposition 2.9] it is shown that if $(X, \mathscr{B}, \mu)$ is a standard probability space, and $\mathrm{Aut}(X, \mathscr{B}, \mu)$ is endowed with the strong operator topology, then the set of transformations that are weakly mixing and rigid is a generic set. Since any rigid automorphism has singular spectrum, we see that the set of singular automorphisms is generic. Now let $\mathcal{S} \subseteq \mathrm{Aut}(X, \mathscr{B}, \mu)$ denote the collection of strongly mixing transformation, and note that $\mathcal{S}$ is a meager set since an automorphism cannot simultaneously be rigid and strongly mixing. Since $\mathcal{S}$ is not a complete metric space with respect to the topology induced by the strong operator topology, a new topology was introduced in [25], with respect to which $\mathcal{S}$ is a complete metric space. It is shown in the Corollary to Theorem 7 of [25] that a generic $T \in \mathcal{S}$ has singular spectrum, and such a T is mixing of all orders due a well known result of Host [20]. See [11] and [21] for concrete examples of $T \in \mathcal{S}$ that have singular spectrum. See also [1], [8], and [16].

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