Coding of geodesic flow and rigidity/flexibility of entropies for Fuchsian boundary maps

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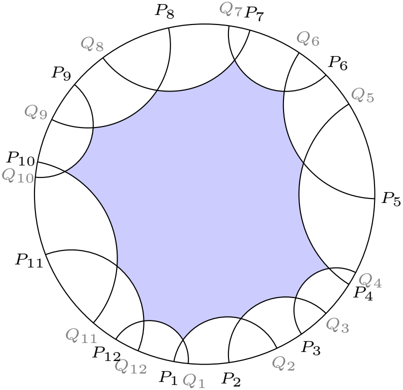
Fundamental polygon

For any genus $g \geq 2$ compact, closed, oriented surface S of constant negative curvature, $S = \Gamma\backslash\mathbb{D}$ for a Fuchsian group $\Gamma \subset \mathrm{Isom}^+(\mathbb{D})$.

- Adler and Flatto describe a fundamental $(8g-4)$-gon $\mathcal{F}$ such that the side-pairing transformations $T_i : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ generate Γ.



Side i extends to geodesic P_iQ_{i+1}.

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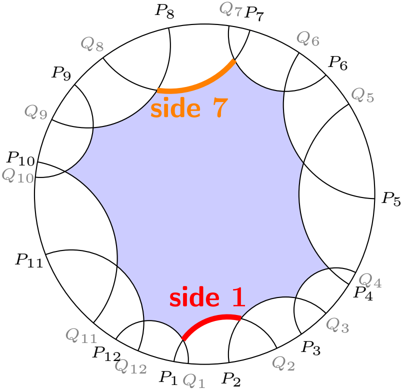

Side i is glued to side $\sigma(i)$ by map T_i.

$$\sigma(i) = \begin{cases} 4g - i & \text{odd } i, \\ 2 - i & \text{even } i. \end{cases}$$

Indices are all $\bmod\ 8g-4$.

Fundamental polygon

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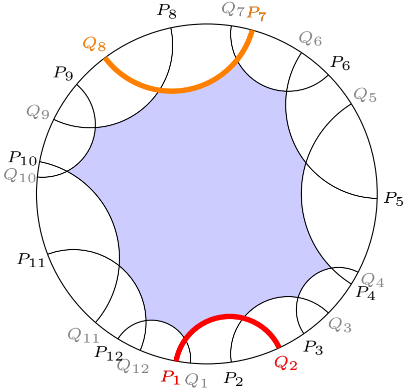
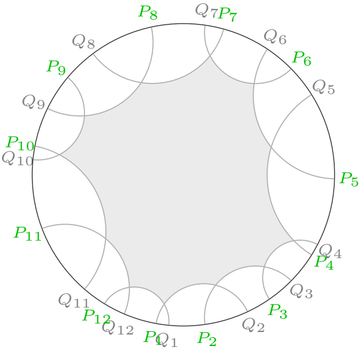


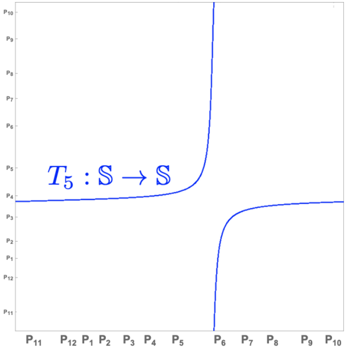
Lemma (Adler–Flatto)

The map T_i sends

$$\begin{aligned} P_{i-1} &\longrightarrow P_{\sigma(i)+1} \\ P_i &\longrightarrow Q_{\sigma(i)+1} \\ Q_i &\longrightarrow Q_{\sigma(i)+2} \\ P_{i+1} &\longrightarrow P_{\sigma(i)-1} \\ Q_{i+1} &\longrightarrow P_{\sigma(i)} \\ Q_{i+2} &\longrightarrow Q_{\sigma(i)} \end{aligned}$$

Boundary map

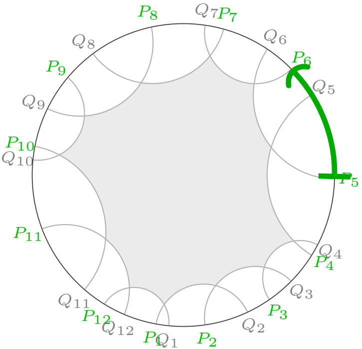


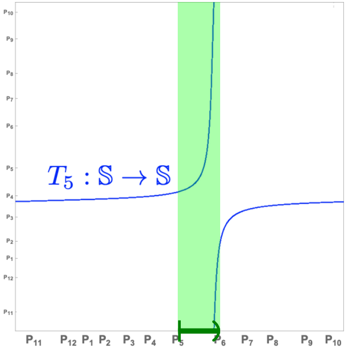


$$\mathbb{S} = \partial\mathbb{D} \cong [-\pi, \pi)$$

Graph: $y = \arg(\frac{ae^{ix}+\bar{c}}{ce^{ix}+\bar{a}})$

Boundary map





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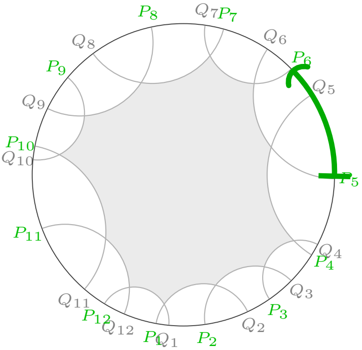
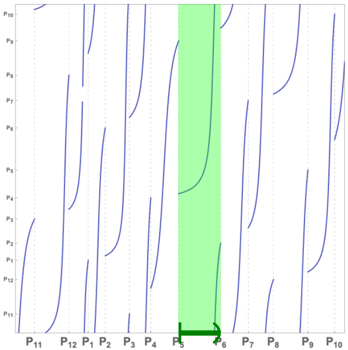


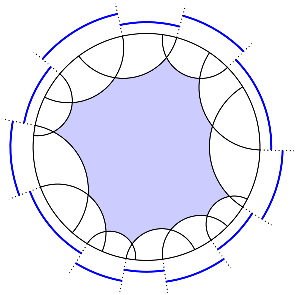


For each fundamental polygon $\mathcal{F}$ with sides along geodesics $P_i Q_{i+1}$, define the "Bowen–Series boundary map" $f_{\bar{P}}$ on $\mathbb{S} = \partial\mathbb{D}$ by

$$f_{\bar{P}}(x) = T_i x \qquad \text{if } x \in [P_i, P_{i+1}).$$

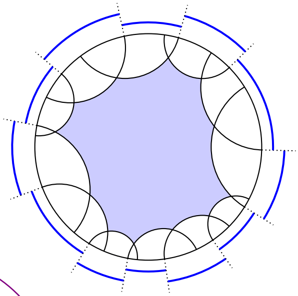
This map has a smooth invariant probability measure $\tilde{\mu}$.

Parameters



Original

Parameters



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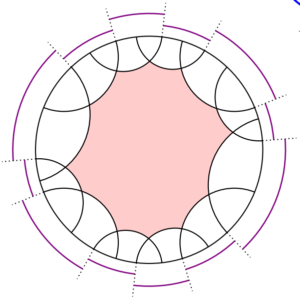
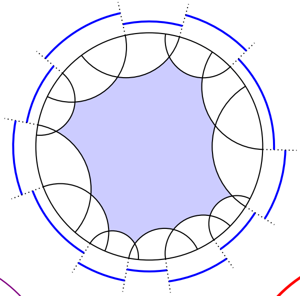


Change polygon

Parameters



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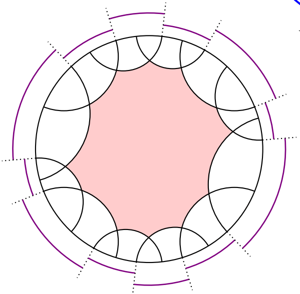
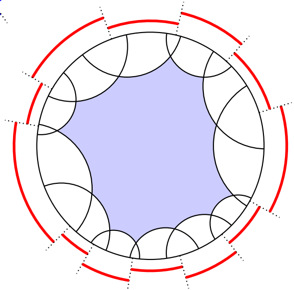


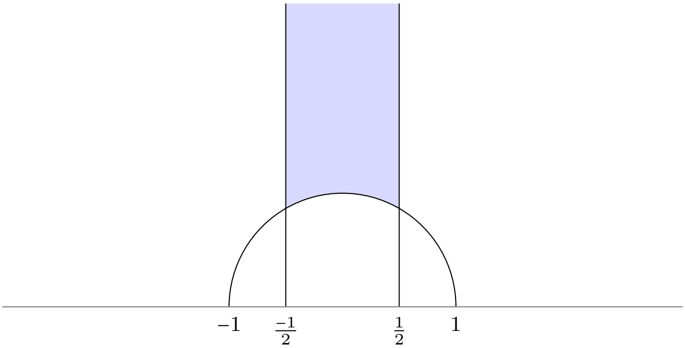
Change polygon



Change interval where each T_i is used

Motivation

The "modular surface" is $M = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}^2$.



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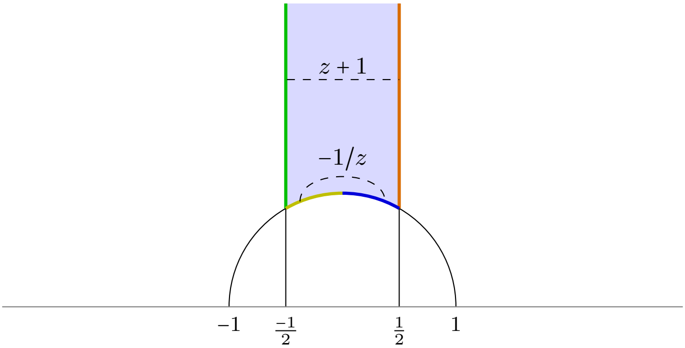

- Geodesic flow on M is related to continued fractions.
- Different continued fraction algorithms use these generators on different intervals of $\mathbb{R}$.

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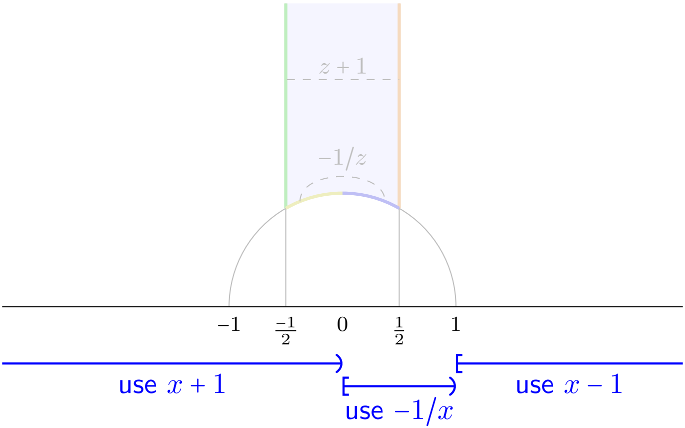

classical continued fraction

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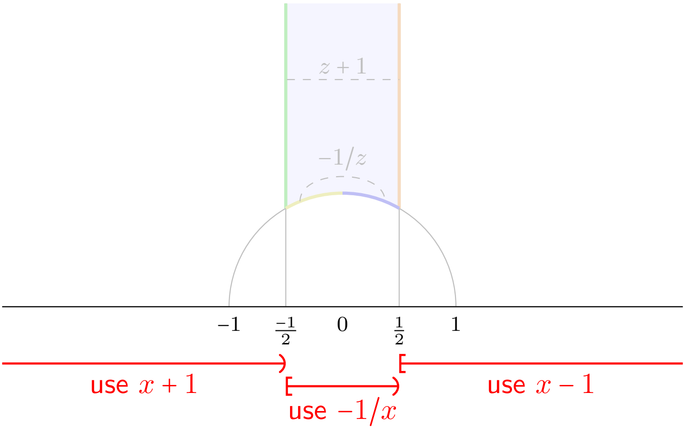

Hurwitz continued fraction, 1880's

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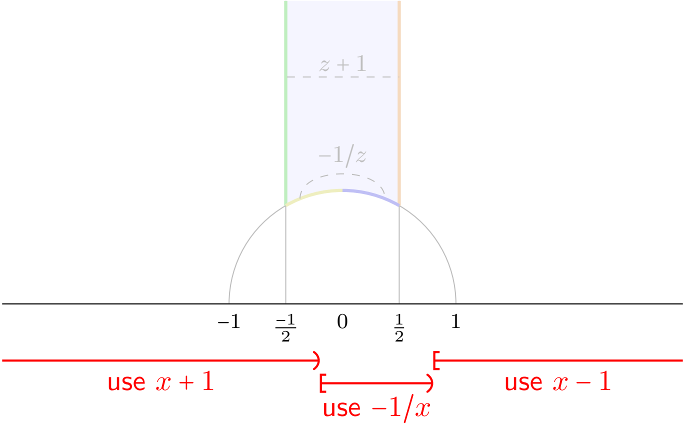

Japanese or α-continued fractions, 2000’s

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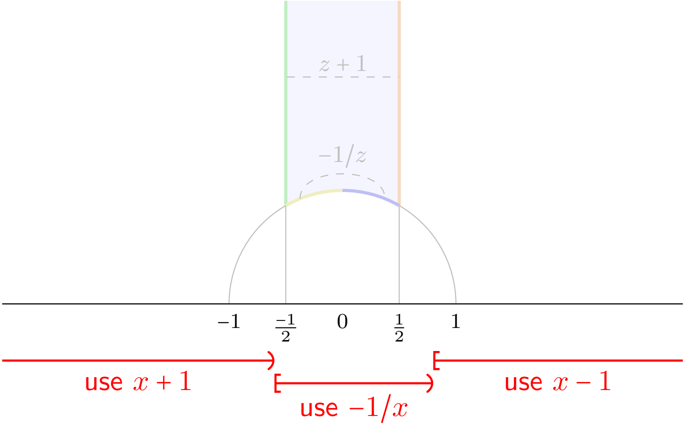

Katok–Ugarcovici (a,b)-continued fractions, 2010's

Classes of parameters

Fix the polygon $\mathcal{F}$. For any parameter choice

$$\bar{A} = \{A_1, A_2, \ldots, A_{8g-4}\}$$

with $A_i \in [P_i, Q_i]$, we can define the boundary map

$$f_{\bar{A}}(w) = T_i w \quad \text{if } x \in [A_i, A_{i+1})$$

Definition

- If each $A_i \in \{P_i, Q_i\}$, then $\bar{A}$ is called extremal.
- If each $A_i \in (P_i, Q_i)$ and $f_{\bar{A}}(T_i A_i) = f_{\bar{A}}(T_{i-1} A_i)$ for all i, then the parameter choice $\bar{A}$ has the short cycle property.

Adler–Flatto studied only $\bar{A} = \{P_1, \ldots, P_{8g-4}\}$ and $\bar{A} = \{Q_1, \ldots, Q_{8g-4}\}$.

Natural extension

The map $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$ is highly non-invertible.

[1] S. Katok, I. Ugarcovici. *Structure of attractors for boundary maps associated to Fuchsian groups*, Geometriae Dedicata **191** (2017), 171–198.

[2] A. Abrams. *Extremal parameters and dual codes for Fuchsian boundary maps*, Illinois Journal of Math.

Natural extension

The map $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$ is highly non-invertible.
The map $F_{\bar{A}}$ on $\mathbb{S} \times \mathbb{S} \setminus \Delta$, where $\Delta = \{ (x, x) : x \in \mathbb{S} \}$, given by

$$F_{\bar{A}}(u, w) = (T_i u, T_i w) \quad \text{if } w \in [A_i, A_{i+1})$$

is also not invertible.

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Theorem

If $\bar{A}$ has the short cycle property [1] or is extremal [2] then there exists $\Omega_{\bar{A}} \subset \mathbb{S} \times \mathbb{S}$ such that

- The restriction $F_{\bar{A}}\big|_{\Omega_{\bar{A}}}$ is bijective.
- The set $\Omega_{\bar{A}}$ has a "finite rectangular structure" and is the global attractor of $F_{\bar{A}}$, that is, $\Omega_{\bar{A}} = \bigcap_{n=0}^{\infty} F_{\bar{A}}^n(\mathbb{S} \times \mathbb{S} \setminus \Delta)$.

The map $F_{\bar{A}} : \Omega_{\bar{A}} \to \Omega_{\bar{A}}$ is the natural extension of $f_{\bar{A}}$.

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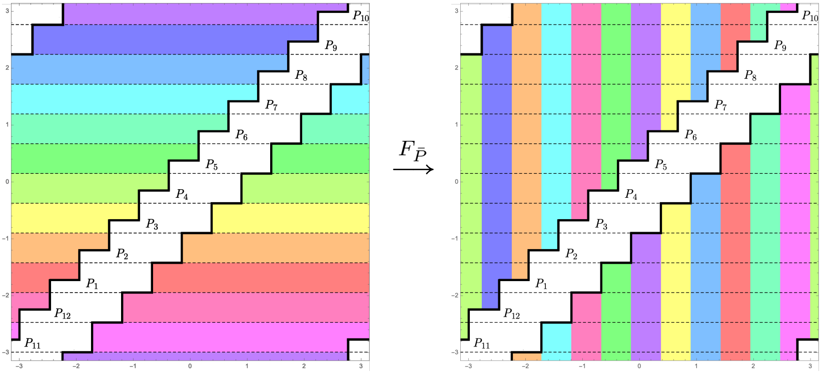
Natural extension

$$F_{\bar{P}}(u,w) = (T_i u, T_i w) \quad \text{if } w \in [P_i, P_{i+1})$$



This domain $\Omega_{\bar{P}}$ is called the "arithmetic set".

Natural extension

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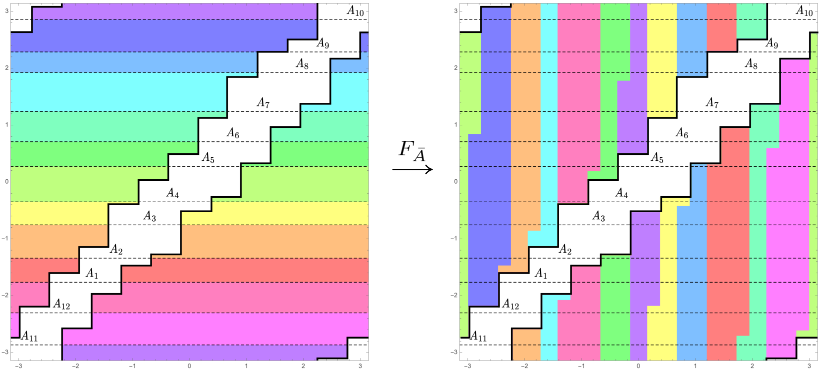


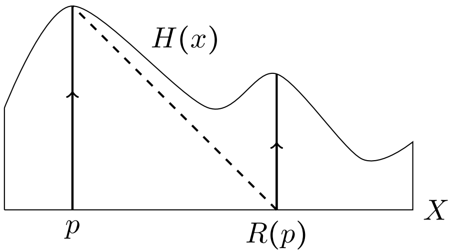
This domain $\Omega_{\bar{A}}$ is called the "arithmetic set".

Special flows

A special flow over $R : X \to X$ is vertical flow on the space

$$\{\,(x,t) : x \in X, 0 \leq t < H(x)\,\}/\!\sim$$

where $H : X \to \mathbb{R}_+$ and $(x, H(x)) \sim (R(x), 0)$.



A cross-section of a flow φ^t on M is a subset $C \subset M$ to which almost every orbit returns infinitely often. The flow φ^t is a special flow over $R : C \to C$ with $H(x)$ the "first return time" and $R(x) = \varphi^{H(x)}(x)$.

Special flows

We have two maps we can iterate:

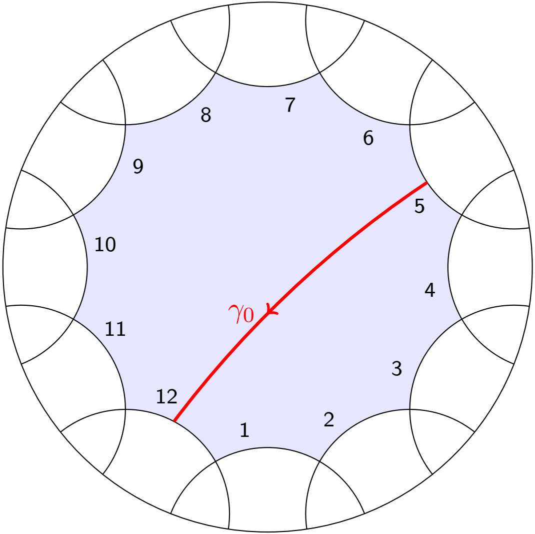
- $f_{\bar{A}}$ on the circle $\mathbb{S}$.
- $F_{\bar{A}}$ on the domain $\Omega_{\bar{A}} \subset \mathbb{S} \times \mathbb{S}$.

The goal is to show that geodesic flow on $S = \Gamma \backslash \mathbb{D}$ is a special flow over $F_{\bar{A}}$ and then use this to produce results about $f_{\bar{A}}$.

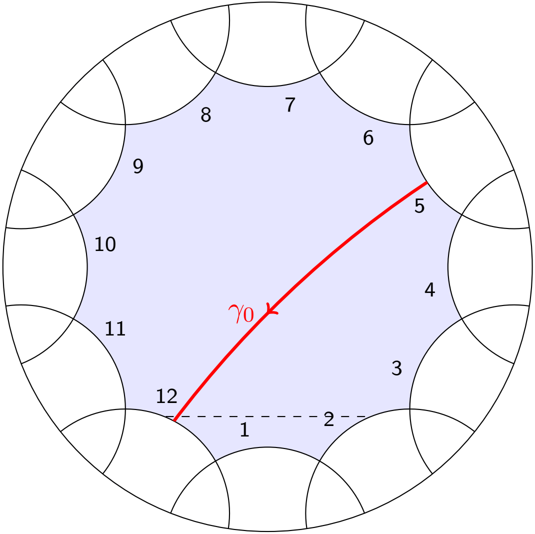
- This requires constructing an "arithmetic cross-section" $C_{\bar{A}}$.

There is already a natural "geometric cross-section" for geodesic flow.

Geodesic flow



Geodesic flow



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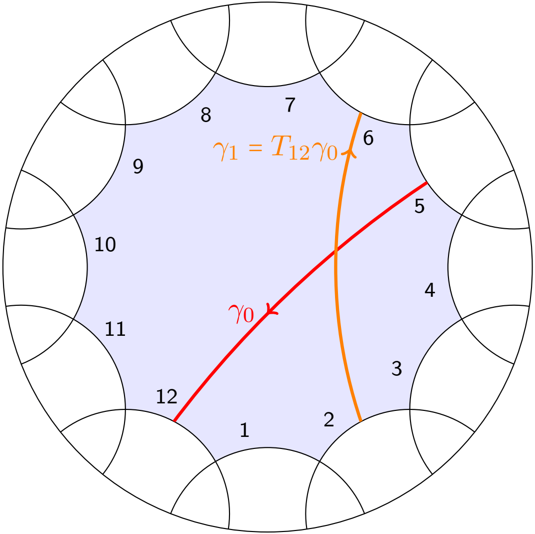

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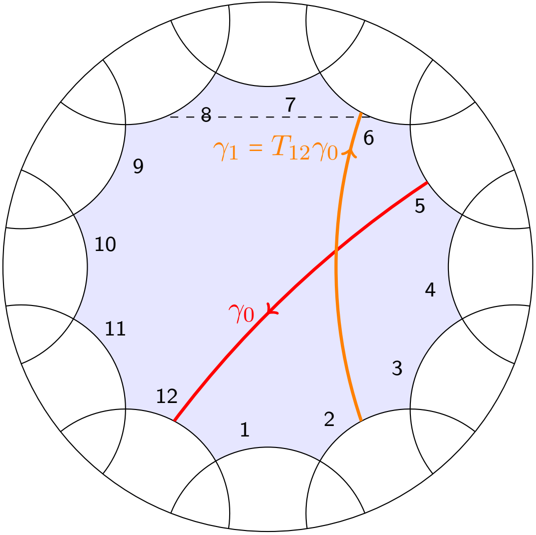

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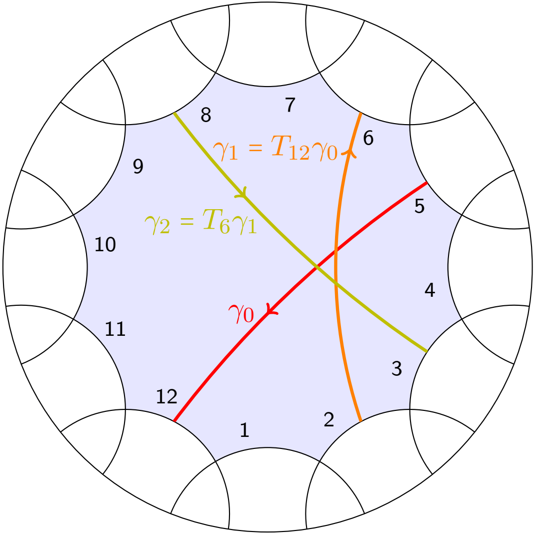

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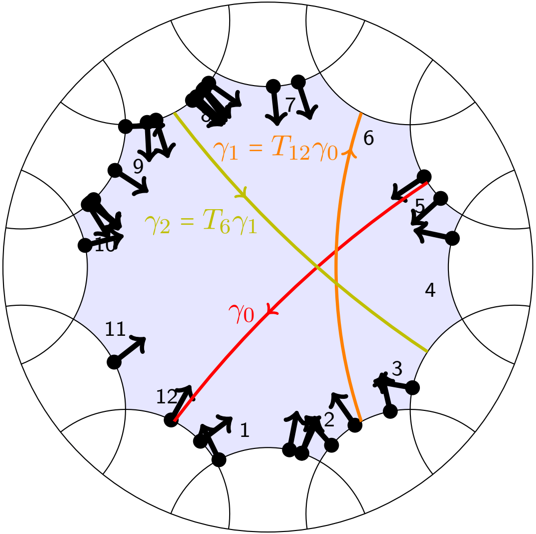

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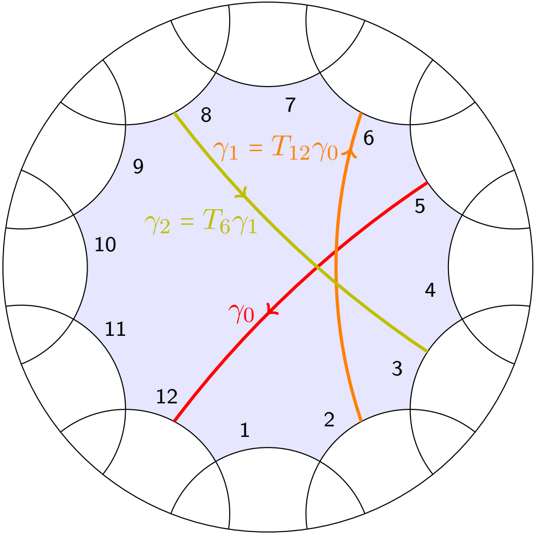

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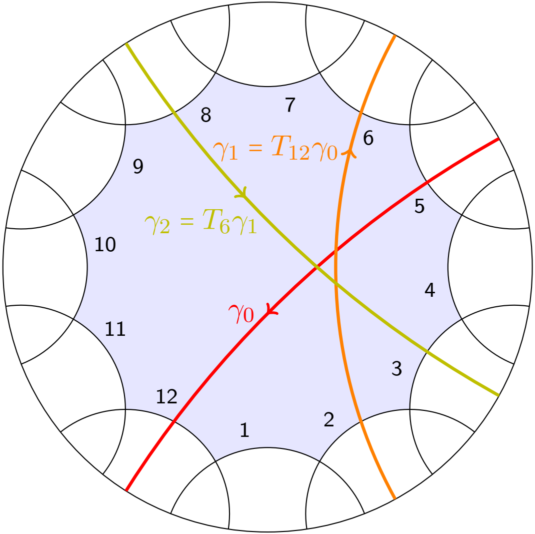

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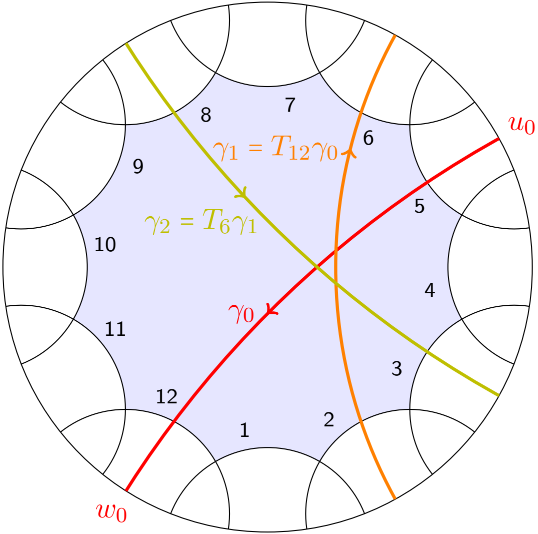
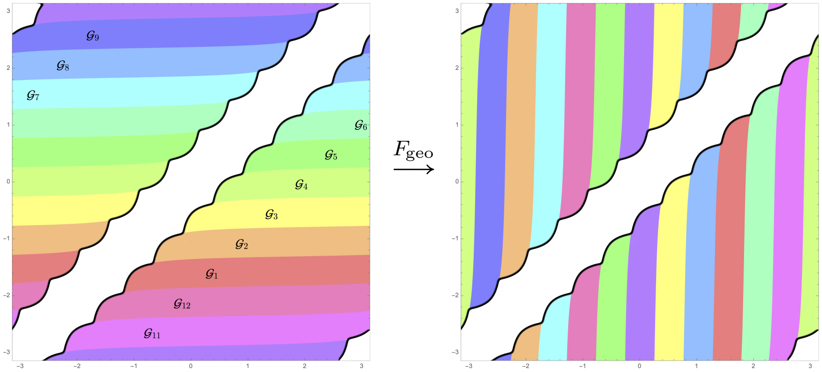


Geometric map

$$F_{\text{geo}}(u,w) = (T_i u, T_i w) \quad \text{if } uw \text{ exits } \mathcal{F} \text{ through side } i$$

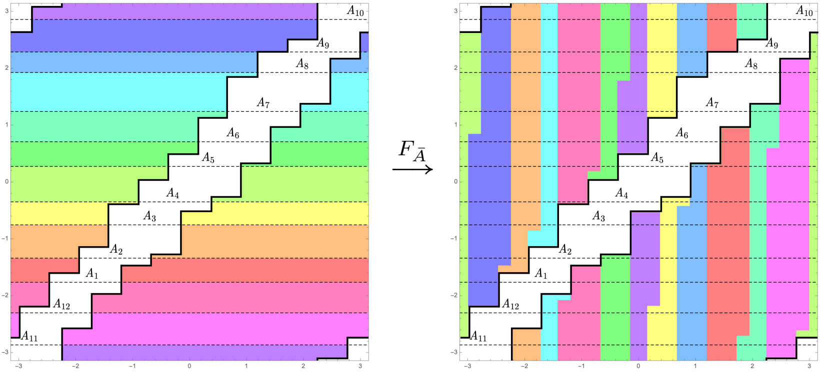


Domain $\Omega_{\text{geo}} = \{ (u,w) : uw \text{ intersects } \mathcal{F} \}$.

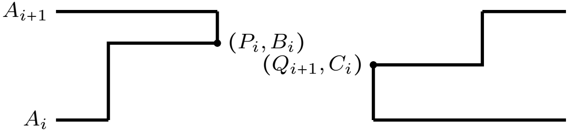
By construction, geo. flow is a special flow over $F_{\text{geo}} : \Omega_{\text{geo}} \to \Omega_{\text{geo}}$.

Arithmetic map

$$F_{\bar{A}}(u, w) = (T_i u, T_i w) \quad \text{if } w \in [A_i, A_{i+1})$$



Structure of attractor

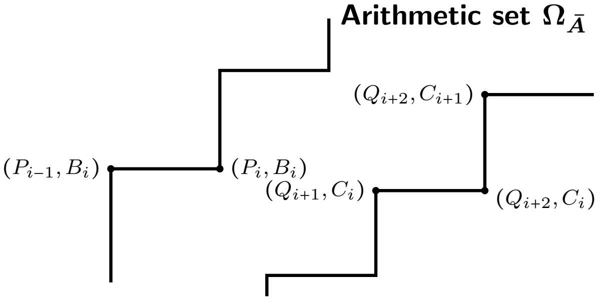


The corner points are

$$\text{upper part: } (P_i, B_i) \quad \text{and} \quad \text{lower part: } (Q_{i+1}, C_i),$$

where $B_i := T_{\sigma(i-1)} A_{\sigma(i-1)}$ and $C_i := T_{\sigma(i+1)} A_{\sigma(i+1)+1}$.

Structure of attractor



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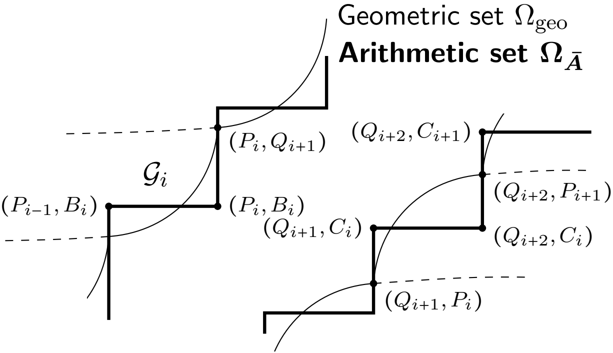

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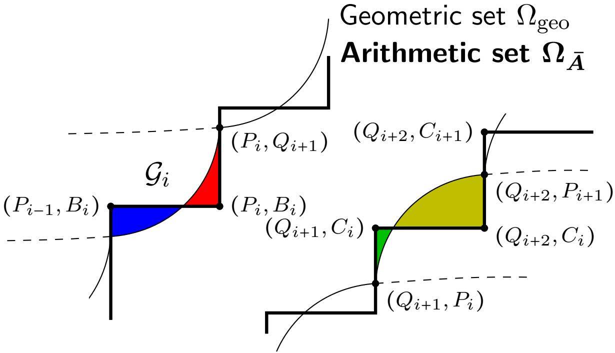

upper corner $\mathcal{C}^i$ lower bulge $\mathcal{B}_i$

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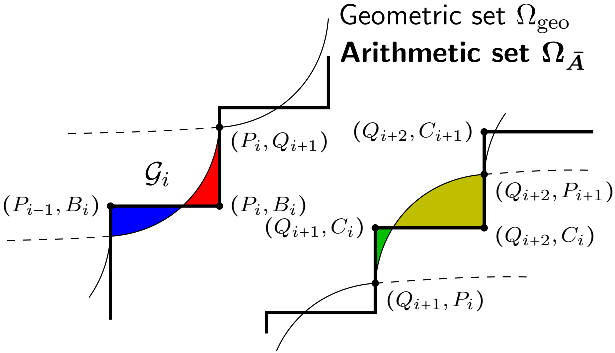
overlap $\mathcal{O} = \Omega_{\text{geo}} \cap \Omega_{\bar{A}}$

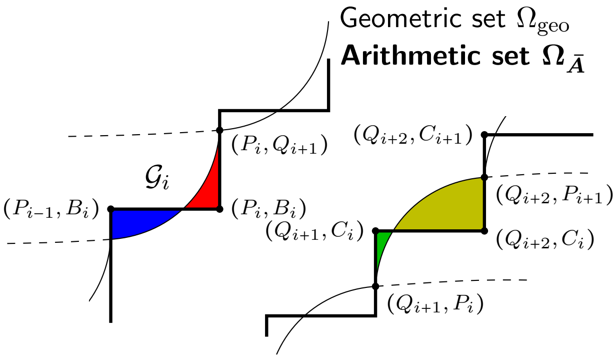
Structure of attractor



$$\Omega_{\text{geo}} = \mathcal{O} \cup \bigcup_{i=1}^{8g-4} \mathcal{B}^i \cup \mathcal{B}_i$$

$$\Omega_{\bar{A}} = \mathcal{O} \cup \bigcup_{i=1}^{8g-4} \mathcal{C}^i \cup \mathcal{C}_i$$

Structure of attractor



Goal: construct

$$\Phi : \Omega_{\text{geo}} \to \Omega_{\bar{A}}$$

such that

$$\Phi \circ F_{\text{geo}} = F_{\bar{A}} \circ \Phi$$

Notation

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- New permutation

$$\tau(i) := i + (4g - 2).$$

P_i and $P_{\tau(i)}$ are antipodal. Q_i and $Q_{\tau(i)}$ are antipodal.

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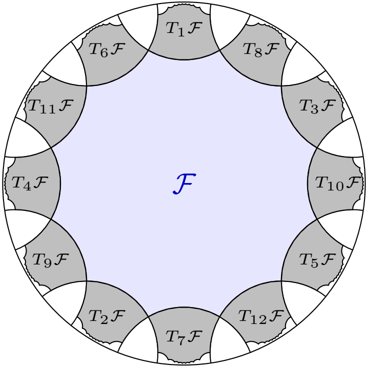
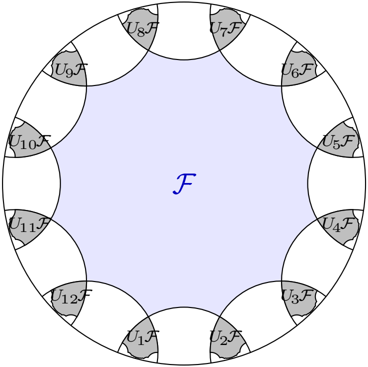
Lemma

- $\sigma(i - 1) = \tau\sigma(i) + 1.$
- $U_i^{-1} = U_{\tau(i)}.$
- $U_i\mathcal{F}$ touches $\mathcal{F}$ at vertex i (where sides i and $i-1$ meet).

Circle maps





The map

$$U_i = T_{\sigma(i)}T_{\tau(i-1)} = T_{\sigma(i-1)}T_{\tau(i)}$$

sends $\mathcal{F}$ to the "corner image" $U_i\mathcal{F}$.

Arithmetic vs. geometric

Proposition (A.-Katok)

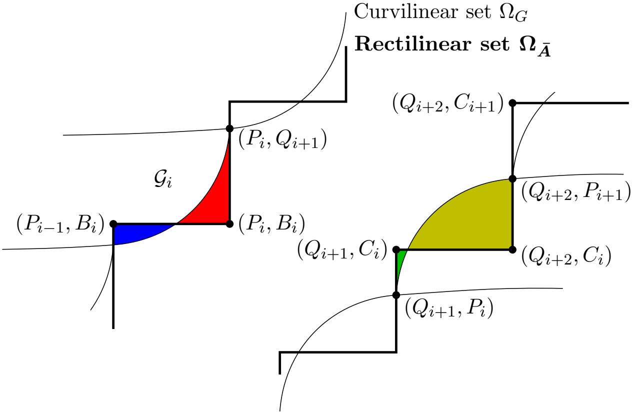
Let $\bar{A}$ have the short cycle property, and let $\mathcal{B}_i, \mathcal{C}_i, \mathcal{B}^i, \mathcal{C}^i$ be the bulges and corners shown previously. The map Φ with domain Ω_{geo} given by

$$\Phi = \begin{cases} \mathrm{Id} & \text{on } \mathcal{O} \\ U_{\tau(i)+1} & \text{on } \mathcal{B}_i \\ U_{\tau(i)} & \text{on } \mathcal{B}^i \end{cases}$$

is a bijection from Ω_{geo} to $\Omega_{\bar{A}}$.

Specifically, $\Phi(\mathcal{B}_i) = \mathcal{C}^{\tau(i)+1}$ and $\Phi(\mathcal{B}^i) = \mathcal{C}_{\tau(i)-1}$.

Arithmetic vs. geometric



$$\Phi = \begin{cases} \mathrm{Id} & \text{on } \mathcal{O} \\ U_{\tau(i)+1} & \text{on } \mathcal{B}_i \\ U_{\tau(i)} & \text{on } \mathcal{B}^i \end{cases}$$

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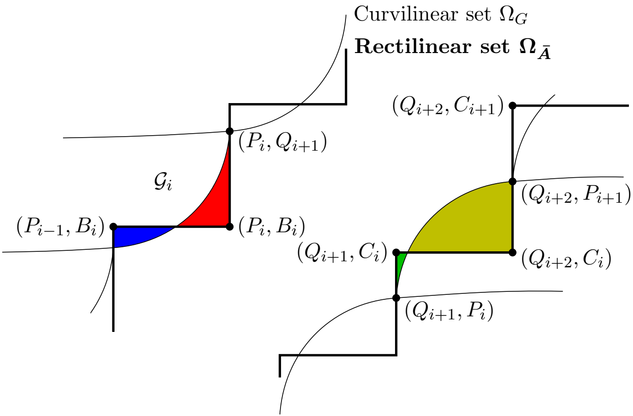
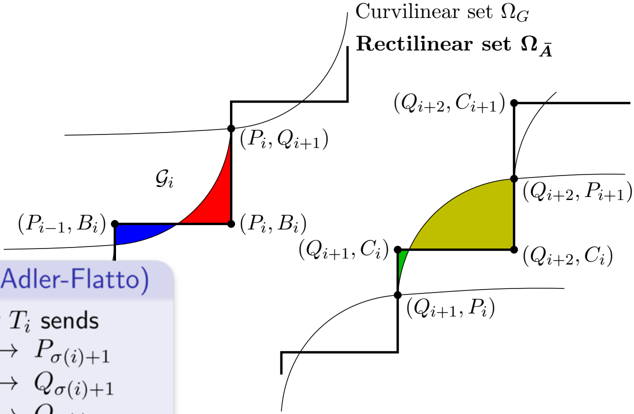


$$U_{\tau(i)+1}\mathcal{B}_i = \mathcal{C}^{\tau(i)+1}$$

$$U_{\tau(i)}\mathcal{B}^i = \mathcal{C}_{\tau(i)-1}$$

Arithmetic vs. geometric



Lemma (Adler-Flatto)

Generator T_i sends

$$\begin{array}{lcl} P_{i-1} & \longrightarrow & P_{\sigma(i)+1} \\ P_i & \longrightarrow & Q_{\sigma(i)+1} \\ Q_i & \longrightarrow & Q_{\sigma(i)+2} \\ P_{i+1} & \longrightarrow & P_{\sigma(i)-1} \\ Q_{i+1} & \longrightarrow & P_{\sigma(i)} \\ Q_{i+2} & \longrightarrow & Q_{\sigma(i)} \end{array}$$

$$U_{\tau(i)+1}\mathcal{B}_i = \mathcal{C}^{\tau(i)+1}$$

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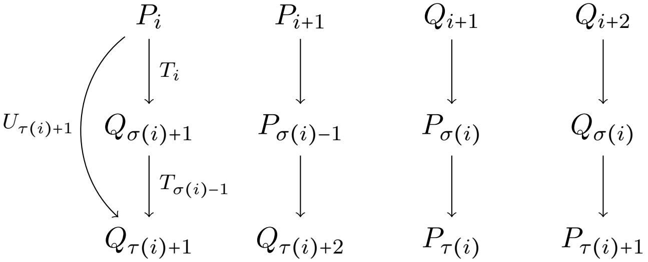
Arithmetic vs. geometric

$$\begin{array}{cccc}
P_i & P_{i+1} & Q_{i+1} & Q_{i+2} \\
\Big\downarrow{\scriptstyle T_i} & \Big\downarrow & \Big\downarrow & \Big\downarrow \\
Q_{\sigma(i)+1} & P_{\sigma(i)-1} & P_{\sigma(i)} & Q_{\sigma(i)}
\end{array}$$

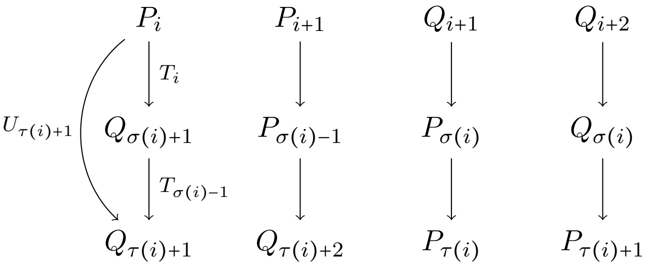
Arithmetic vs. geometric

$$
\begin{array}{cccc}
P_i & P_{i+1} & Q_{i+1} & Q_{i+2} \\
\Big\downarrow{\scriptstyle T_i} & \Big\downarrow & \Big\downarrow & \Big\downarrow \\
Q_{\sigma(i)+1} & P_{\sigma(i)-1} & P_{\sigma(i)} & Q_{\sigma(i)} \\
\Big\downarrow{\scriptstyle T_{\sigma(i)-1}} & \Big\downarrow & \Big\downarrow & \Big\downarrow \\
Q_{\tau(i)+1} & Q_{\tau(i)+2} & P_{\tau(i)} & P_{\tau(i)+1}
\end{array}
$$

Arithmetic vs. geometric



Arithmetic vs. geometric



and

$$\begin{aligned} U_{\tau(i)+1}C_i &= (T_{\sigma(i+1)+1}T_{i+1})(T_{\sigma(i+1)}A_{\tau\sigma(i)}) \\ &= T_{\sigma(\tau(i))}A_{\sigma(\tau(i))} \\ &= B_{\tau(i)+1} \end{aligned}$$

Conjugacy

Theorem (A.-Katok)

Let $\bar{A}$ satisfy the short cycle property. Then $\Phi : \Omega_{\text{geo}} \to \Omega_{\bar{A}}$ is a conjugacy between F_{geo} and $F_{\bar{A}}$. That is, the following diagram commutes:

$$\begin{array}{ccc} \Omega_{\text{geo}} & \xrightarrow{F_{\text{geo}}} & \Omega_{\text{geo}} \\ \Phi \downarrow & & \downarrow \Phi \\ \Omega_{\bar{A}} & \xrightarrow[F_{\bar{A}}]{} & \Omega_{\bar{A}} \end{array}$$

[3] A. Abrams, S. Katok. *Adler and Flatto revisited: cross-sections for geodesic flow on compact surfaces of constant negative curvature*, Studia Mathematica **246** (2019), 167–202.

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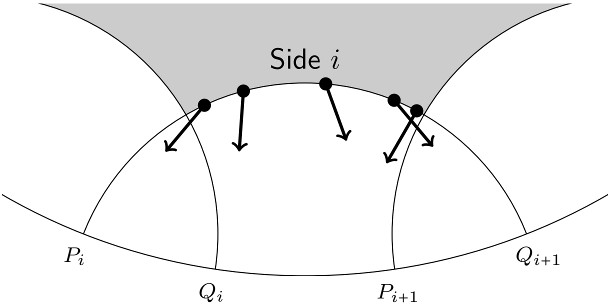
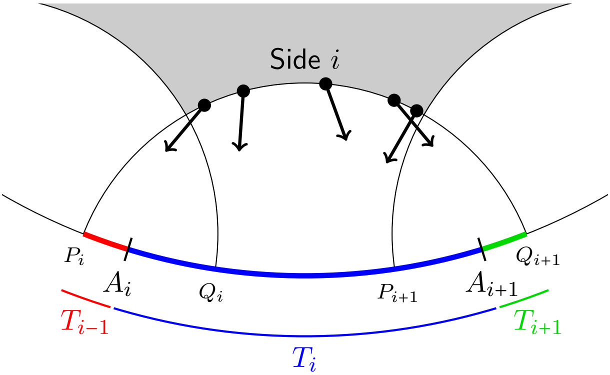
Proof method: Look how the four maps in $\begin{array}{ccc} (u, w) & \xrightarrow{F_{\text{geo}}} & \\ \Phi \downarrow & & \downarrow \Phi \\ & \xrightarrow[F_{\bar{A}}]{} & \end{array}$ act.

[3] A. Abrams, S. Katok. *Adler and Flatto revisited: cross-sections for geodesic flow on compact surfaces of constant negative curvature*, Studia Mathematica **246** (2019), 167–202.

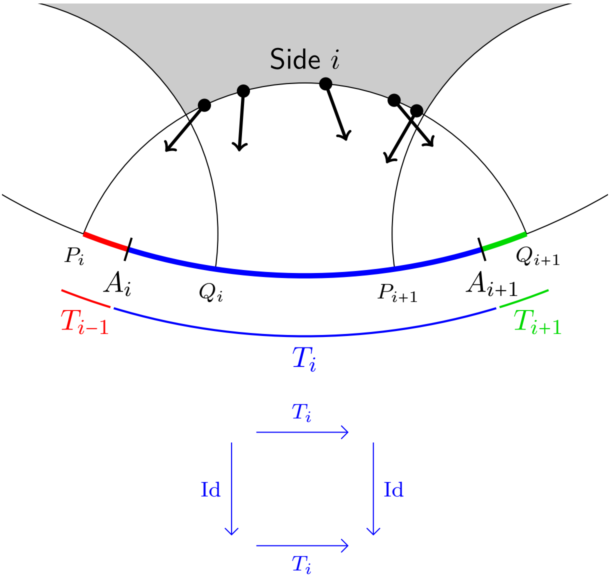
Conjugacy



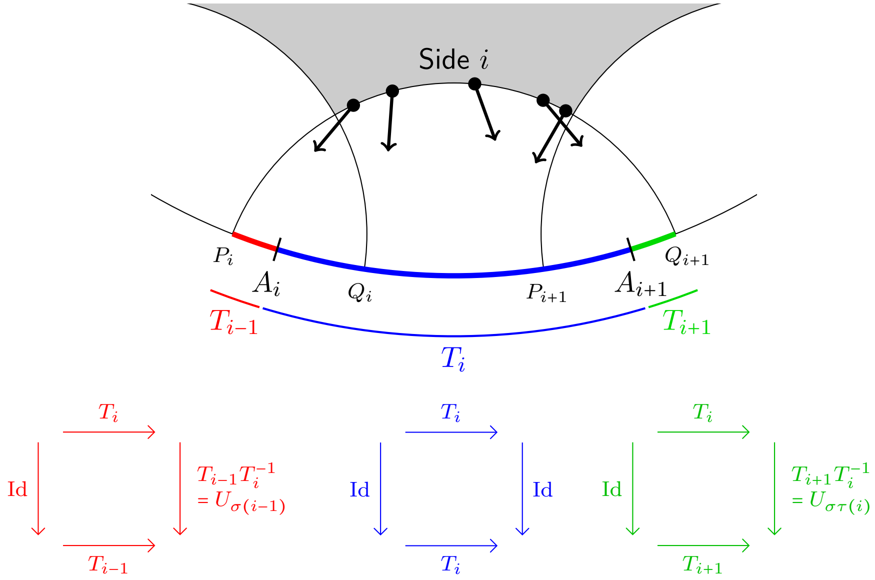
Conjugacy



Conjugacy



Conjugacy


Side i
P_i
A_i
Q_i
P_{i+1}
A_{i+1}
Q_{i+1}
T_{i-1}
T_i
T_{i+1}
T_i
Id
$T_{i-1}T_i^{-1}$
$= U_{\sigma(i-1)}$
T_{i-1}
T_i
Id
Id
T_i
T_i
Id
$T_{i+1}T_i^{-1}$
$= U_{\sigma\tau(i)}$
T_{i+1}

Cross-section

Because $\Phi : \Omega_{\text{geo}} \to \Omega_{\bar{A}}$ bijectively, we know that if $(u,w) \in \Omega_{\bar{A}}$ then the geodesic $\gamma = uw$ intersects $\mathcal{F}$ or intersects $U_j\mathcal{F}$, where

$$j = \begin{cases} i & \text{if } (u,w) \in \mathcal{C}^i \\ i+1 & \text{if } (u,w) \in \mathcal{C}_i. \end{cases}$$

Definitions

- A geodesic $\gamma = uw$ is called reduced if $(u,w) \in \Omega_{\bar{A}}$.
- The cross-section point of a reduced geodesic γ is the point where it enters $\mathcal{F}$ or $U_j\mathcal{F}$.
- The arithmetic cross-section is

$$C_{\bar{A}} = \left\{ \pi(z,\zeta) \;\middle|\; \begin{array}{c} z \text{ is the cross-section point} \\ \text{of a reduced geodesic } \gamma, \\ \zeta \text{ is tangent to } \gamma \text{ at } z \end{array} \right\}$$

where $\pi : T^1\mathbb{D} \to T^1 S$ is projection.

Arithmetic coding

Given any $w \in \mathbb{S}$, we can build a sequence $n_0, n_1, n_2, \ldots$ by recording which interval each $w_k = f_{\bar{A}}^k(w)$ is in.

- This gives only a one-sided sequence, but geodesic flow can move forwards or backwards.
- Recall that $f_{\bar{A}}$ is not invertible but $F_{\bar{A}}\big|_{\Omega_{\bar{A}}}$ is.

Arithmetic coding

Let $\gamma = uw$ be a reduced geodesic on $\mathbb{D}$, and denote

$$(u_k, w_k) = F_{\bar{A}}^k(u, w) \qquad \text{for all } k \in \mathbb{Z}.$$

Definition

The arithmetic code of $\gamma = uw$ is the sequence

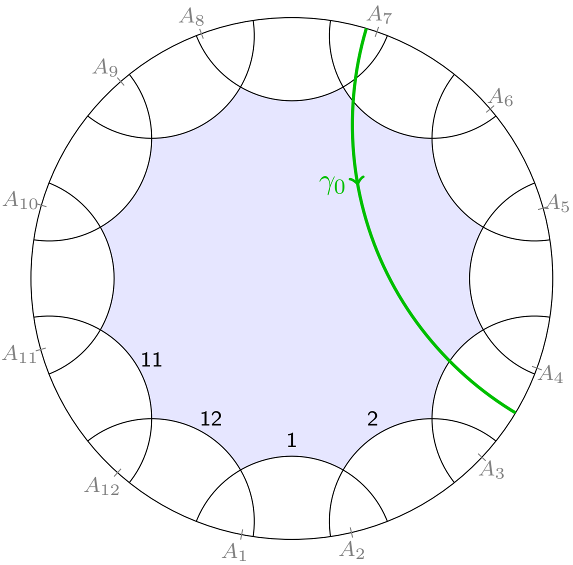
$$[\gamma]_{\bar{A}} = (\ldots, n_{-2}, n_{-1}, n_0, n_1, n_2, \ldots)$$

where $n_k = \sigma(i)$ for the index i such that $w_k \in [A_i, A_{i+1})$.

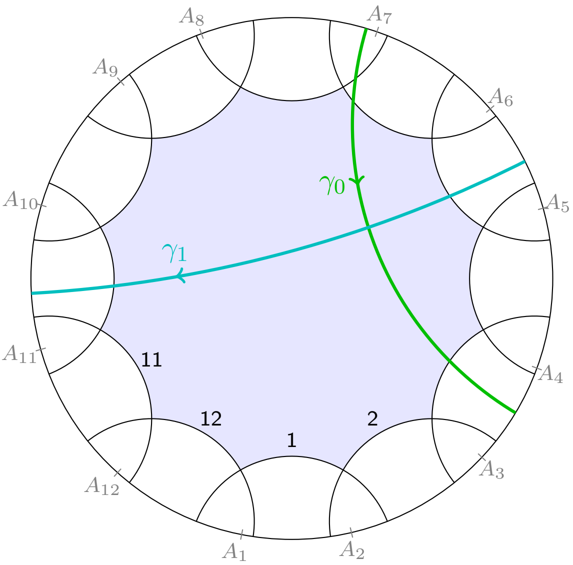
Theorem (A.-Katok)

Let $\bar{\gamma}$ be the projection of γ to $S = \Gamma\backslash\mathbb{D}$. Then the first return of the flow along $\bar{\gamma}$ to the cross-section $C_{\bar{A}}$ corresponds to a left shift of the coding sequence $[\gamma]_{\bar{A}}$.

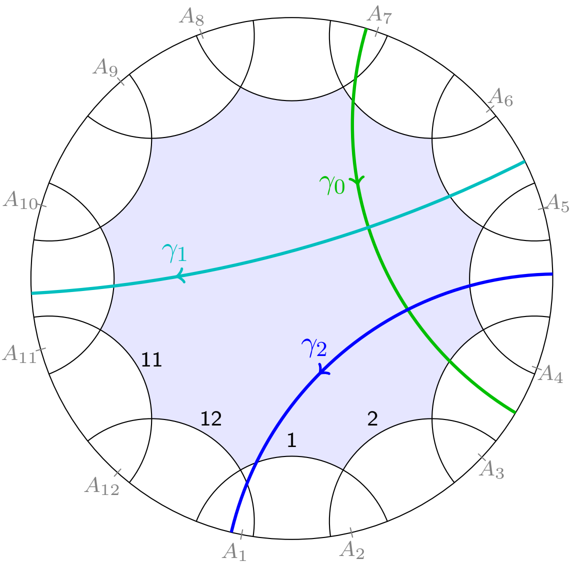
Coding example



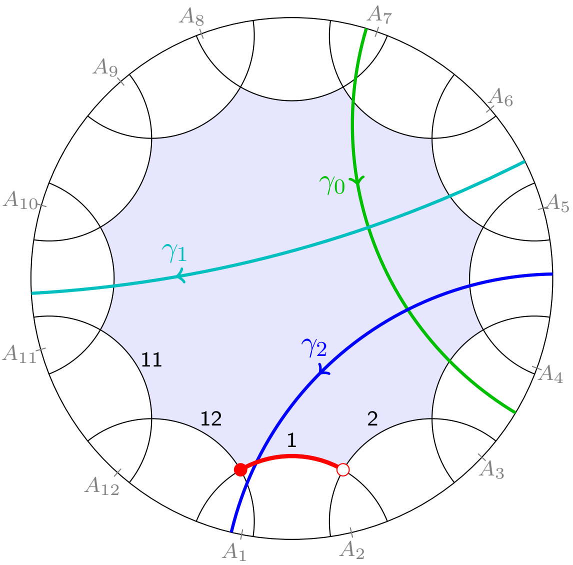
Coding example



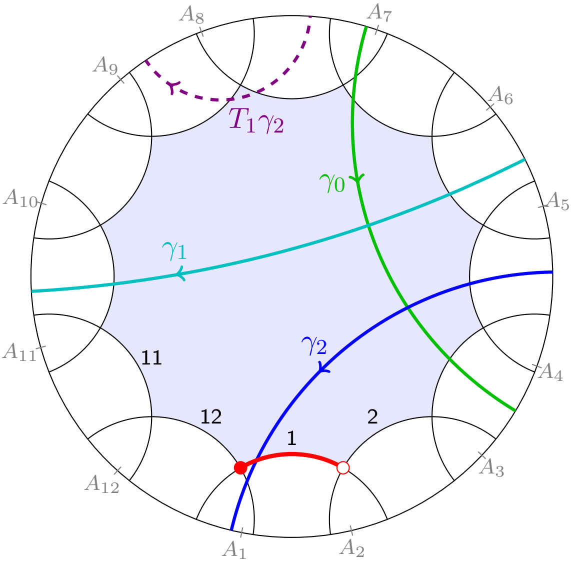
Coding example



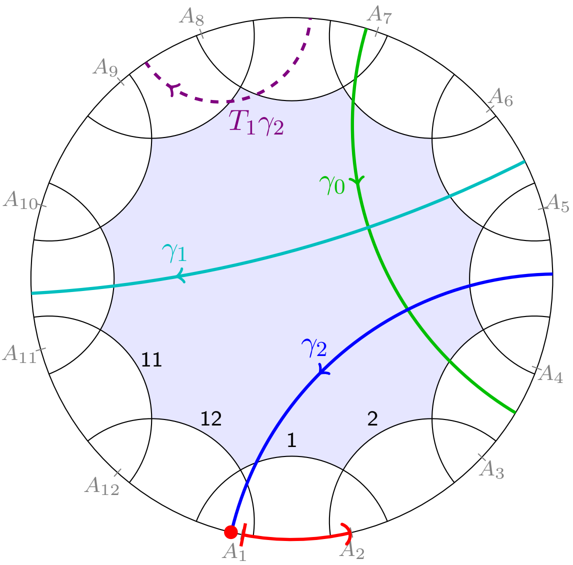
Coding example



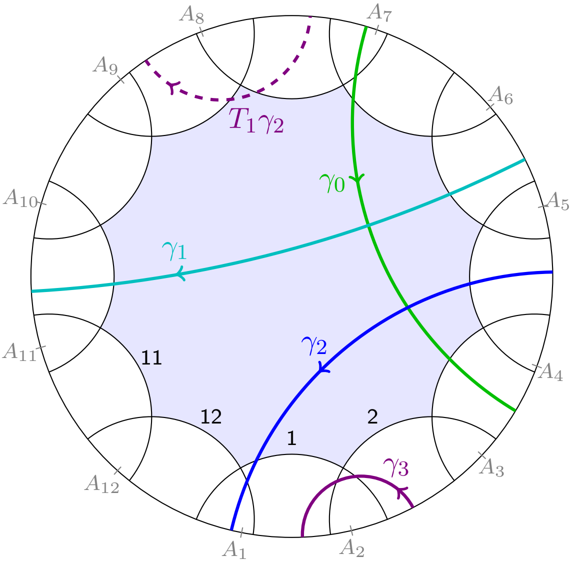
Coding example



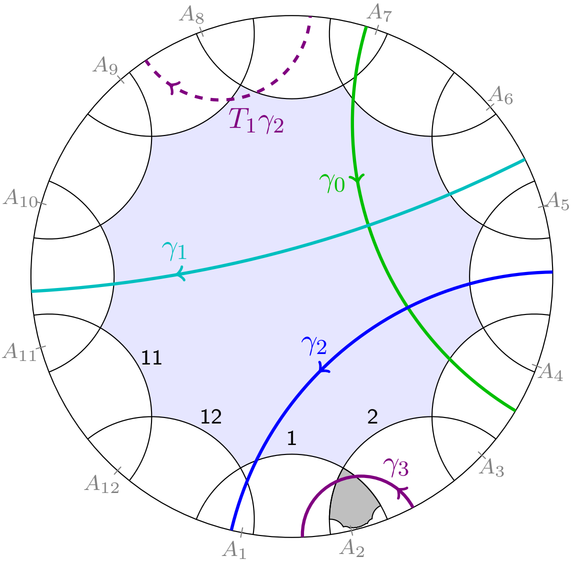
Coding example



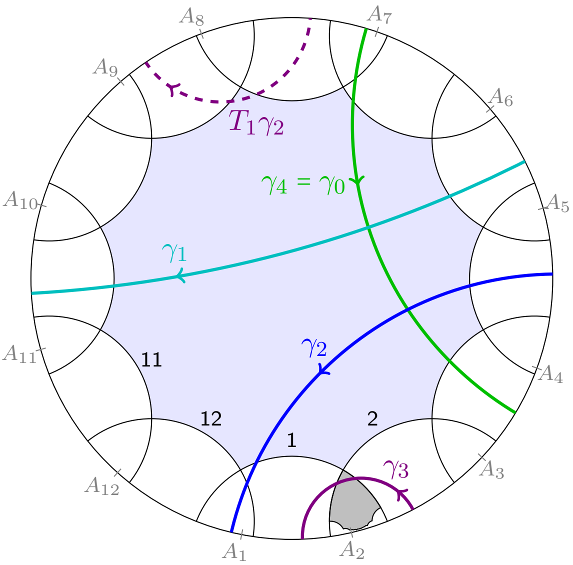
Coding example



Coding example



Coding example



Coding example

A periodic code $(\ldots, n_k, n_0, n_1, \ldots, n_{k-1}, n_k, n_0, n_1, \ldots)$ is written as just $(n_0, \ldots, n_k)$.

- Let γ be the axis of $T_5 T_4 T_7 T_6$.
    - Its geometric code is
    $$[\gamma]_{\text{geo}} = (\sigma(3), \sigma(10), \sigma(1), \sigma(8)) = (5, 4, 7, 6).$$
    - Its arithmetic code is
    $$[\gamma]_{\bar{A}} = (\sigma(3), \sigma(10), \sigma(12), \sigma(1)) = (5, 4, 2, 7).$$

Coding example

A periodic code $(..., n_k, n_0, n_1, ..., n_{k-1}, n_k, n_0, n_1, ...)$ is written as just $(n_0, ..., n_k)$.

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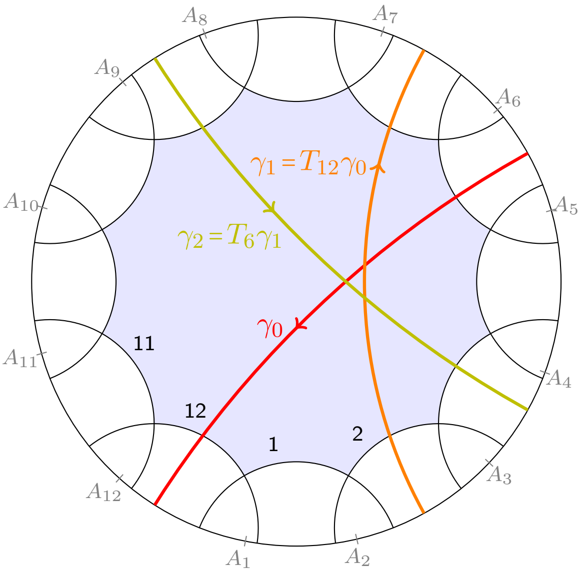
- For the axis of $T_2 T_8 T_5$,
$$[\gamma]_{\text{geo}} = [\gamma]_{\bar{A}} = (\sigma(12), \sigma(6), \sigma(3)) = (2, 8, 5).$$

Coding example



Dual codes

Recall the coding sequence of $\gamma = uw$ is

$$[\gamma]_{\bar{A}} = (\ldots, n_{-2}, n_{-1}, n_0, n_1, n_2, \ldots)$$

where

$$n_k = \sigma(i) \quad \text{if } w_k \in [A_i, A_{i+1}),$$

and $(u_k, w_k) = F^k_{\bar{A}}(u, w)$.

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- Since $f_{\bar{A}}(x) = F_{\bar{A}}(\cdot, x)$, we also have $w_k = f_{\bar{A}}^k(w)$.
- The "future" $n_0, n_1, n_2, \dots$, depends only on w and can be calculated using the one-dimensional map $f_{\bar{A}}$.
- But the "past" $(k < 0)$ generally depends on both u and w.

Dual codes

Definition

Let $\phi(x,y) = (y,x)$. We say $\bar{A}$ and $\bar{B}$ are dual if $\phi(\Omega_{\bar{A}}) = \Omega_{\bar{B}}$ and $\phi(F_{\bar{A}}^{-1}(p)) = F_{\bar{B}}(\phi(p))$ for all $p = (u,w) \in \Omega_{\bar{A}}$ with $u \notin \bar{B}$.

$$\begin{array}{ccc} \Omega_{\bar{A}} & \xrightarrow{F_{\bar{A}}^{-1}} & \Omega_{\bar{A}} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \Omega_{\bar{B}} & \xrightarrow[F_{\bar{B}}]{} & \Omega_{\bar{B}} \end{array}$$

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Theorem (A.-Katok)

If $\bar{A}$ and $\bar{B}$ are dual and $(u,w) \in \Omega_{\bar{A}}$, then the arithmetic code

$$[\gamma]_{\bar{A}} = (\dots, n_{-2}, n_{-1}, n_0, n_1, n_2, \dots)$$

of the geodesic $\gamma = uw$ satisfies

- for $k \geq 0$, $n_k = \sigma(i)$ such that $f_{\bar{A}}^k(w) \in [A_i, A_{i+1})$;
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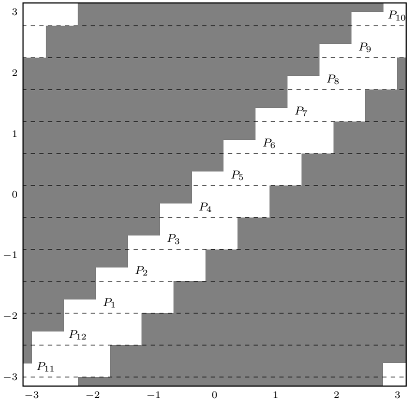
Proposition

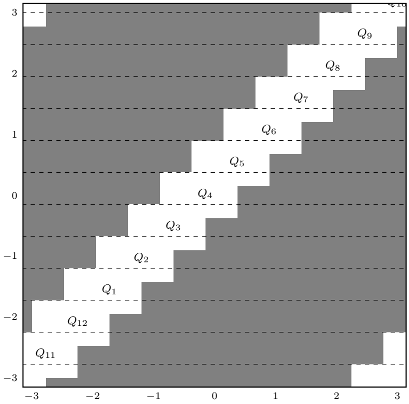
There do not exist dual $\bar{A}$ and $\bar{B}$ with the short cycle property.

Dual example

$\bar{A} = \bar{P} = \{P_1, P_2, P_3, \ldots, P_{12}\}$



$\bar{A} = \bar{Q} = \{Q_1, Q_2, Q_3, \ldots, Q_{12}\}$



Extremal parameters

Recall that a parameter choice $\bar{A} = \{A_1, ..., A_{8g-4}\}$ is called extremal if each $A_i \in \{P_i, Q_i\}$.

Theorem (A.)

For each extremal parameter choice $\bar{A}$ there exists a parameter choice $\bar{B} = \{B_1, ..., B_{8g-4}\}$ such that $\bar{A}$ and $\bar{B}$ are dual.

Previous results described the structure of $\Omega_{\bar{A}}$ only when $\bar{A}$ has short cycles or the specific cases $\bar{A} = \bar{P}$ and $\bar{A} = \bar{Q}$.

- Before discussing the dual, we first need to describe $\Omega_{\bar{A}}$ for extremal $\bar{A}$.
- The parameters $\bar{B}$ might not be extremal or have short cycles, so the domain $\Omega_{\bar{B}}$ of $F_{\bar{B}}$ also does not follow from previous results.

[2] A. Abrams. *Extremal parameters and dual codes for Fuchsian boundary maps*, Illinois Journal of Math.

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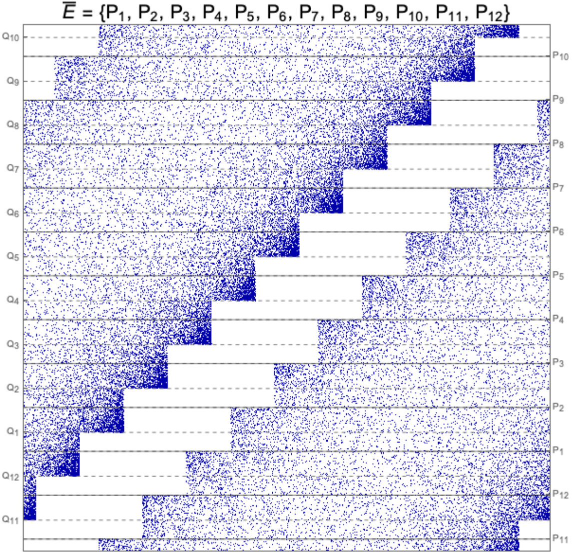
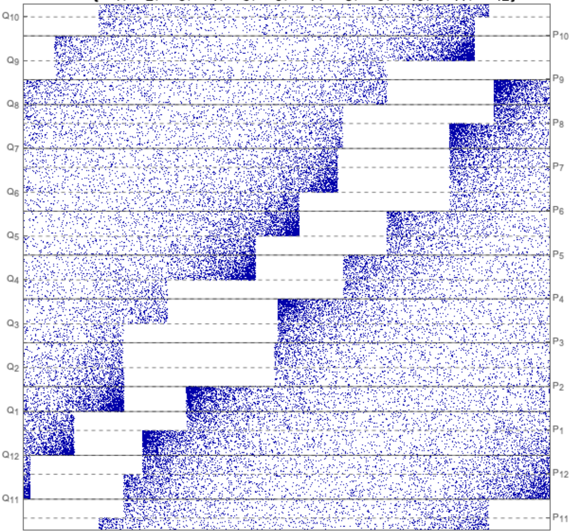
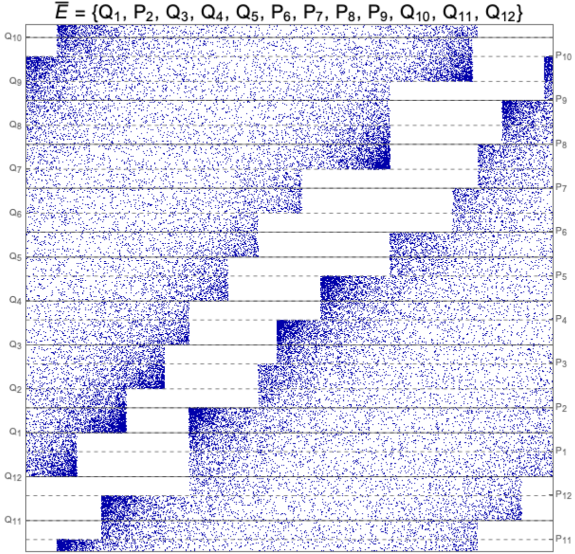


Extremal parameters

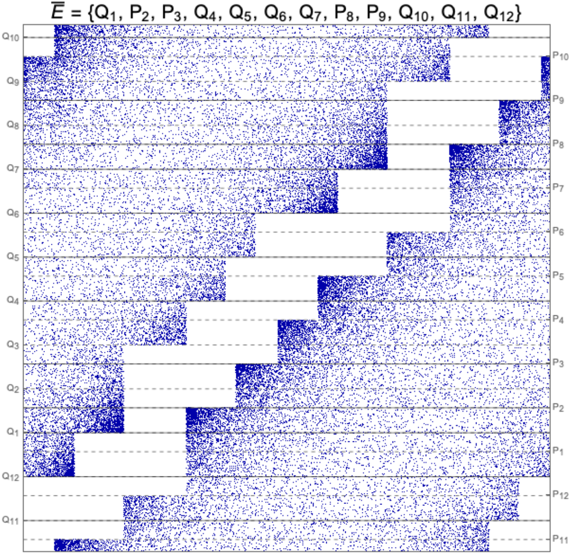
$\overline{E} = \{Q_1, P_2, P_3, P_4, P_5, P_6, Q_7, Q_8, P_9, P_{10}, Q_{11}, Q_{12}\}$



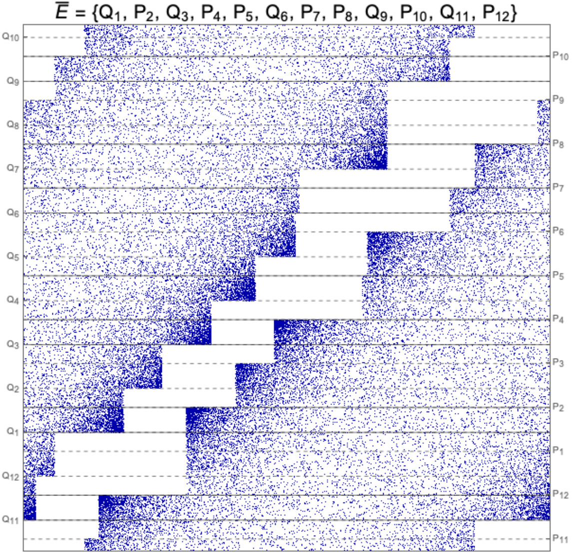
Extremal parameters



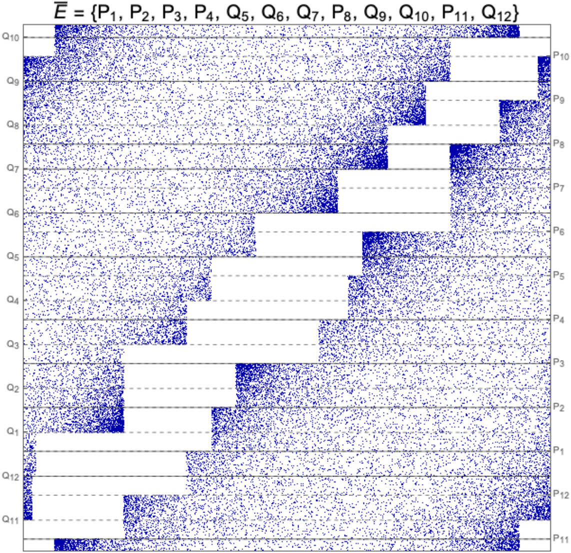
Extremal parameters



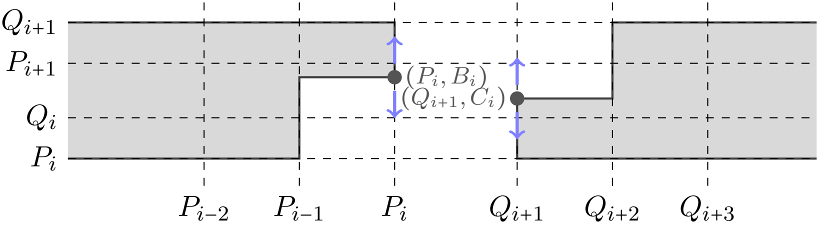
Extremal parameters



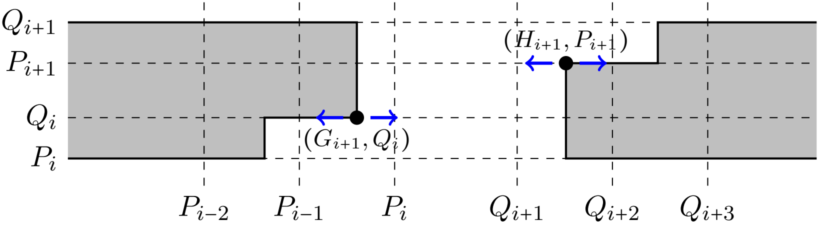
Extremal parameters



Describing the domain



Part of attractor $\Omega_{\bar{A}}$ for short cycles.



Part of attractor $\Omega_{\bar{A}}$ for extremal.

Describing the domain

Suppose a set of the form

$$\Lambda = \bigcup_{i=1}^{8g-4} [H_{i+1}, G_{i-2}] \times [P_i, Q_i] \cup [H_{i+1}, G_{i-1}] \times [Q_i, P_{i+1}]$$

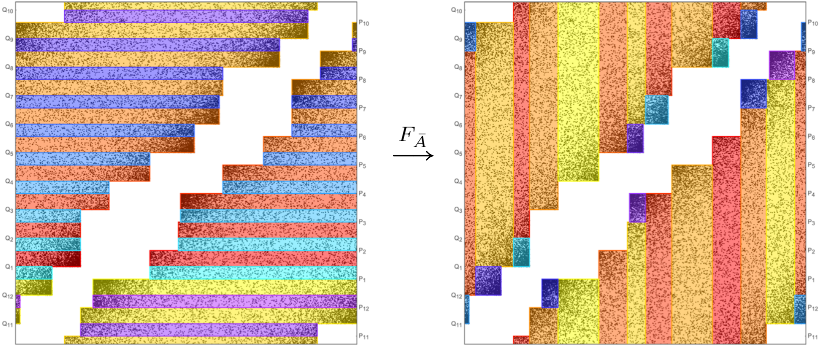
satisfies $F_{\bar{A}}(\Lambda) = \Lambda$ for some extremal $\bar{A}$. What conditions does this imply for $\{G_i\}$ and $\{H_i\}$?

Describing the domain

Suppose a set of the form

$$\Lambda = \bigcup_{i=1}^{8g-4} [H_{i+1}, G_{i-2}] \times [P_i, Q_i] \cup [H_{i+1}, G_{i-1}] \times [Q_i, P_{i+1}]$$

satisfies $F_{\bar{A}}(\Lambda) = \Lambda$ for some extremal $\bar{A}$. What conditions does this imply for $\{G_i\}$ and $\{H_i\}$?



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System of equations

Proposition (A.)

For any $A_1, \ldots, A_{8g-4}$ with $A_i \in \{P_i, Q_i\}$, there exist unique values $G_1, \ldots, G_{8g-4}$ such that for all $1 \le i \le 8g-4$

- $G_i \in [P_i, P_{i+1}]$,
- $G_{\sigma(i)} = T_i G_{i-2}$ if $A_i = P_i$,
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System of equations

Goal: $G_i \in [P_i, P_{i+1}]$ and $G_{\sigma(i)} = \begin{cases} T_i G_{i-2} & \text{if } A_i = P_i \\ T_{\tau(i)+1} G_{\tau(i)} & \text{if } A_i = Q_i \end{cases}$

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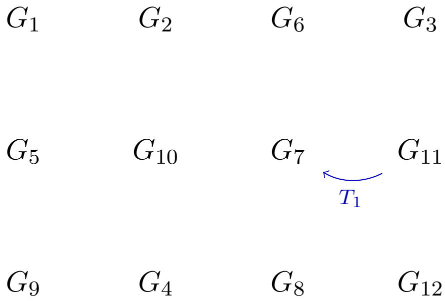

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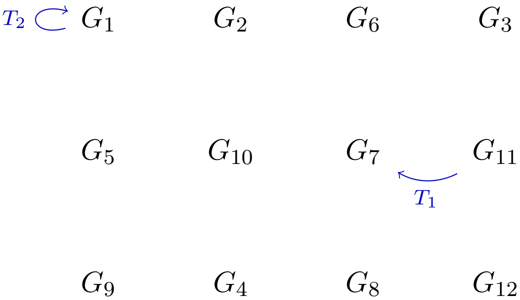

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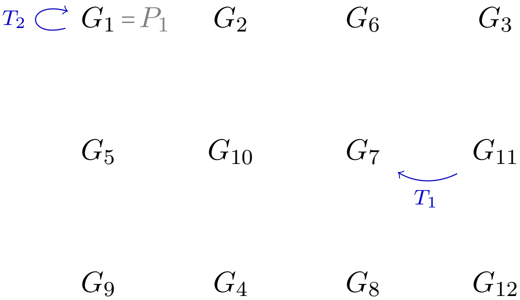

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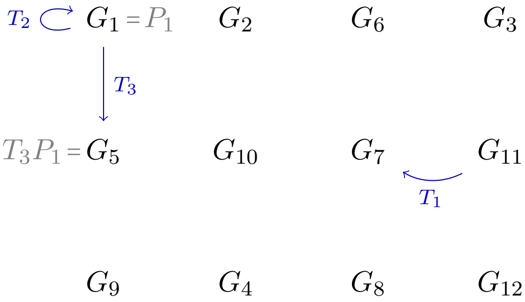

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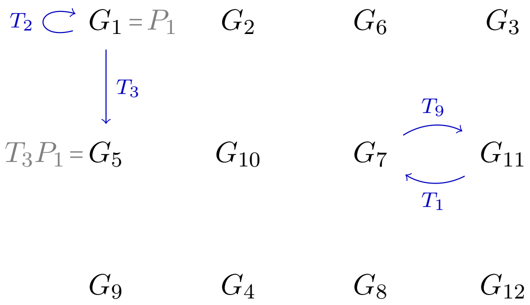

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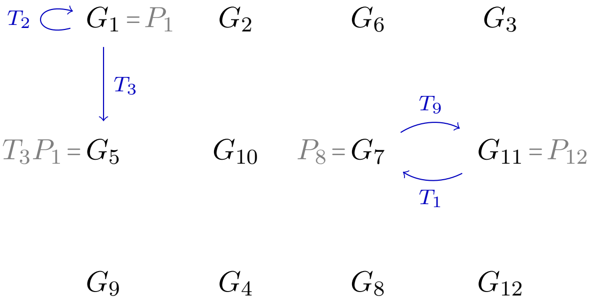

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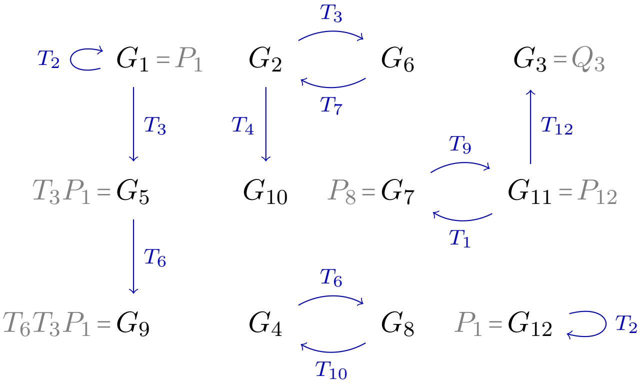

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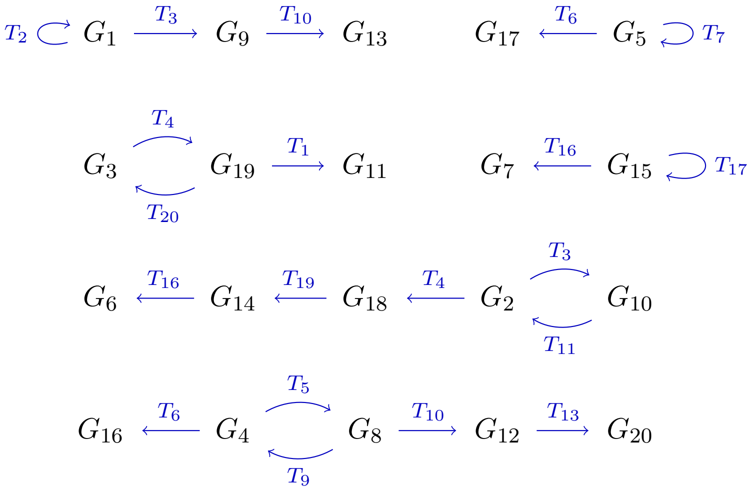

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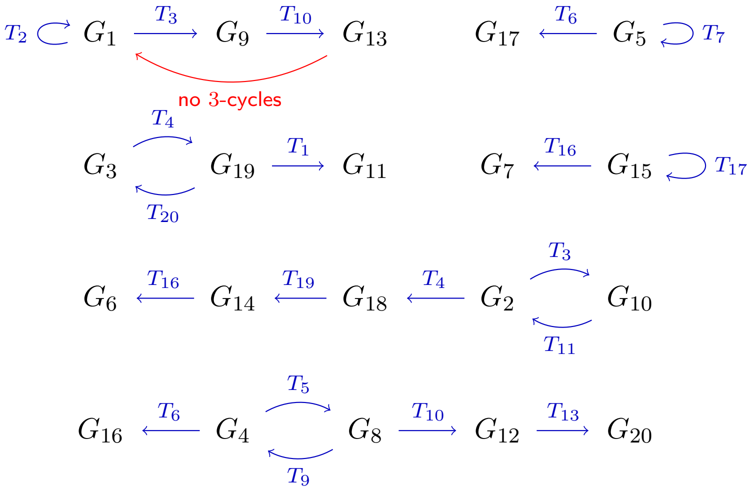

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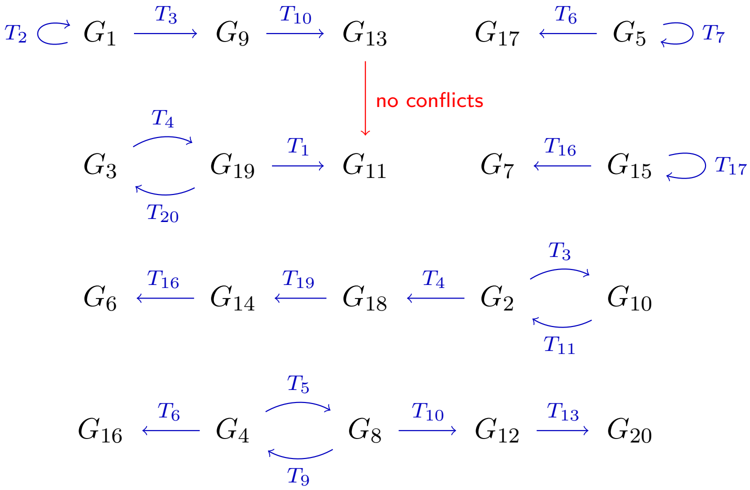

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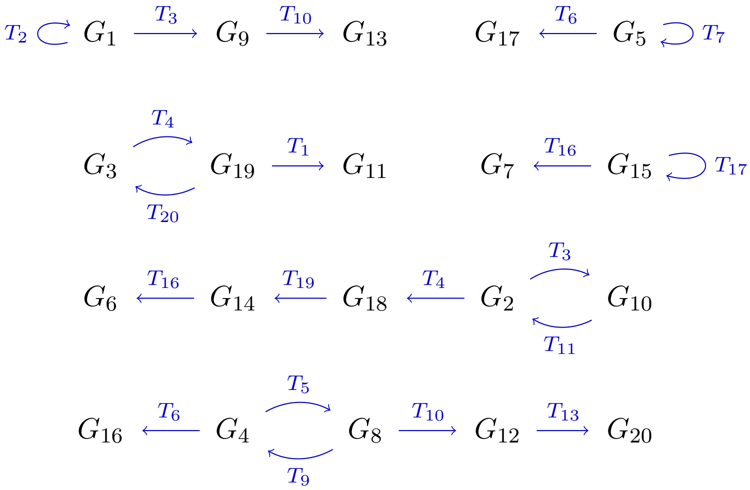

Extremal parameters

Given an extremal $\bar{A}$,

❶ let $G_1, \ldots, G_{8g-4}$ be the unique solution to

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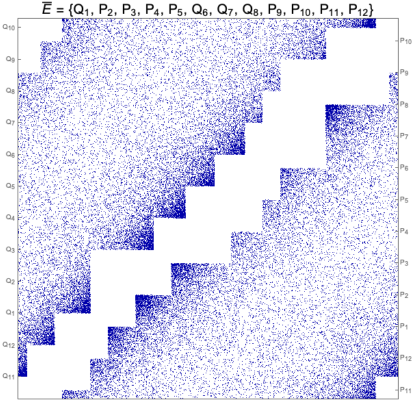
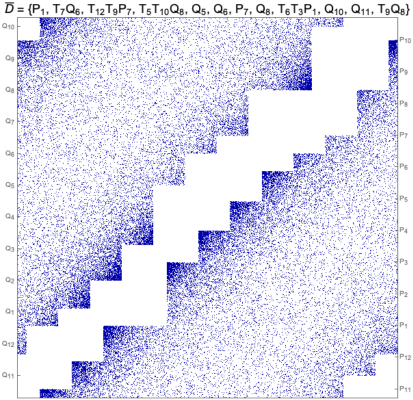
Dual codes





$\bar{A} = \{Q_1, P_2, P_3, P_4, P_5, Q_6, Q_7, Q_8, P_9, P_{10}, P_{11}, P_{12}\}$
is dual to
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Other parameter classes

Domains for $F_{\bar{A}}$ are known when

- $\bar{A}$ satisfies the short cycle property, or
- $\bar{A}$ is extremal, or
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In all these cases, $\Omega_{\bar{A}}$ has finite rectangular structure and $F_{\bar{A}}\big|_{\Omega_{\bar{A}}}$ is conjugate to $F_{\mathrm{geo}} : \Omega_{\mathrm{geo}} \to \Omega_{\mathrm{geo}}$.

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This is *conjectured* to hold for any $\bar{A}$ with $A_i \in [P_i, Q_i]$, but so far we do not even have a clear description of the set $\Omega_{\bar{A}}$ for generic parameters.

Next week

- How can we use $F_{\bar{A}}$ and F_{geo} to compute $h_{\tilde{\mu}}(f_{\bar{A}})$?
- What about $h_{\text{top}}(f_{\bar{P}})$?
- How do these change when we change the parameters $\bar{A}$ or change the polygon $\mathcal{F}$?