Multiorder vs. orbit equivalence to a $\mathbb{Z}$-action

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based on a joint work with

Piotr Oprocha and Guohua Zhang

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some of the ideas presented in this particular section were suggested by Tom Meyerovitch

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The action (1) on total orders is Borel measurable (total orders inherit the Borel structure from $\{0, 1\}^{G \times G}$, the space of all relations in G) and preserves type $\mathbb{Z}$.

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Let $\mathcal{O}$ denote the space of all anchored bijections from $\mathbb{Z}$ to G. Then $\mathcal{O}$ inherits a natural Borel structure from $G^{\mathbb{Z}}$ and the correspondence between total orders of type $\mathbb{Z}$ and bijections from $\mathbb{Z}$ to G is a Borel-measurable bijection.

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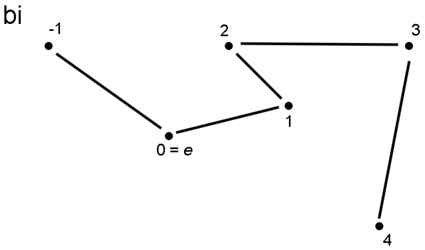

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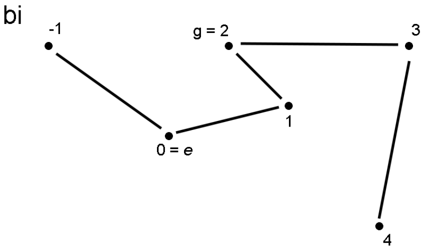

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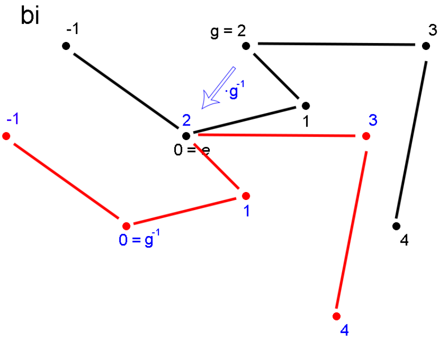

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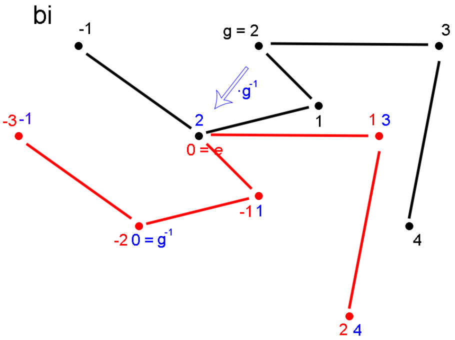

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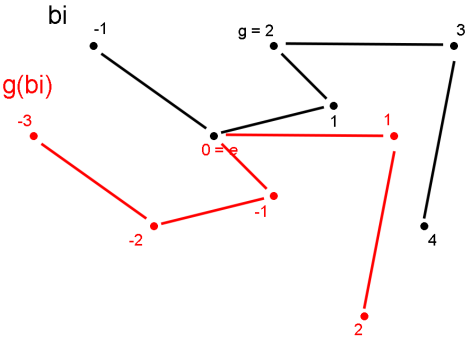

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(Moreover, that multiorder is uniformly Følner, but we will not use this property.)

Orbit equivalence of actions of different groups

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In this case, for μ-almost every $x \in X$ and every $g \in G$ there exists a $\gamma \in \Gamma$ such that

$$\phi(gx) = \gamma(\phi(x)), \tag{3}$$

and every $\gamma \in \Gamma$ satisfies (3) for some g.

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There is usually no uniqueness: more than one element γ may satisfy (3) for given g, one γ may satisfy (3) for more than one g. Uniqueness holds when both actions are free.

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By (4), we have $gx = \gamma x$, (or $\mathrm{id}(gx) = \gamma(\mathrm{id}(x))$), the action (X, μ, Γ) of Γ on (X, μ) defined by (5) is orbit equivalent to the original action (X, μ, G) (with identity playing the role of the conjugating map).

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If, in addition, both actions are free, then for μ-almost every x the correspondence between $g \in G$ and $\gamma \in \Gamma$ given by $gx = \gamma x$ establishes a *bijection* $\mathrm{bi}_x : \Gamma \to G$ (the direction is reversed on purpose).

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Observe that the above bijection is always anchored because $ex = x = e_\Gamma x$.

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Let (X, μ, G) be a free action on a probability space. Let $(X, \mu, \mathbb{Z})$ be a $\mathbb{Z}$-action orbit equivalent (via the identity map) to (X, μ, G). Let $T = T_1$ be the generating map of this $\mathbb{Z}$-action.

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$$\mathrm{bi}_x(i) = g \iff T^i x = gx, \tag{6}$$

is a measure-theoretic factor map from (X, μ, G) to a multiorder $(\mathcal{O}, \nu, G)$, where $\nu = \theta(\mu)$, and the action of G on $\mathcal{O}$ is given by (2).

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Corollary. Since every action of an *amenable* group is orbit-equivalent to a $\mathbb{Z}$-action, every *free* action of an amenable group has a multiorder as a factor.

Multiorder vs. orbit equivalence, (6) $\mathrm{bi}_x(i) = g \iff T^i x = gx$

Proof of Theorem 1. The only thing requiring a proof is the equivariance of the map θ, i.e. we need to show that, for μ-almost all $x \in X$ and all $g \in G$, we have

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By (6) and since the actions are free, the elements $g_1 = \mathrm{bi}_{gx}(i)$ and $g_2 = \mathrm{bi}_x(i+k)$ are (μ-almost surely) the unique members of G for which the respective equalities hold:

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By (6) and since the actions are free, the elements $g_1 = \mathrm{bi}_{gx}(i)$ and $g_2 = \mathrm{bi}_x(i+k)$ are (μ-almost surely) the unique members of G for which the respective equalities hold:

(A) $T^i gx \overset{(6) \text{ applied to } gx}{=} g_1 gx,$

(B) $T^{i+k} x \overset{(6) \text{ applied to } i+k}{=} g_2 x,$

while the fact that $g = \mathrm{bi}_x(k)$ means that

(C) $gx \overset{(6) \text{ applied to } k}{=} T^k x.$

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This is exactly what we needed to show.

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Given $x \in X$, the associated bijection $\mathrm{bi}_x = \varphi(x) \in \mathcal{O}$, and $i \in \mathbb{Z}$, instead of $\mathrm{bi}_x(i)$ we will write i^x (the ith element of G in the order associated to x).
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Let φ be as above. Then (X, μ, G) is orbit-equivalent to the $\mathbb{Z}$-action generated by the *successor map* defined as follows:

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$$S^{k+1}x = S(S^k x) \overset{(8)}{=} S(k^x x) = S(gx) \overset{(7)}{=} 1^{gx}(gx),$$

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Eventually,

$$S^{k+1}(x) = 1^{gx}(gx) = (k + 1)^x g^{-1} gx = (k + 1)^x(x),$$

and (8) is shown for $k + 1$.

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Plugging (11) into (9) we obtain $x = 1^{x'} x' \overset{(7)}{=} S(x') = STx$.

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By a symmetric argument we also have $x = TSx$, which implies, on one hand, that S is invertible (with the inverse T), and on the other hand, that (8) holds for negative integers. This ends the proof. $\square$

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That's all for today!